Original Research Article

Hilbert structure of the Sobolev space $H^1(\Omega)$

Abstract

By studying the Hilbertian structure of the Sobolev space $H^1(\Omega)$, this article highlights the precision and power offered by this mathematical framework. Thanks to the introduction of a specific dot product and the associated completeness property, the Hilbertian structure facilitates the analysis of functions with derivative weaknesses, thus allowing an in-depth study of their convergence, continuity and orthogonality. These characteristics make the Sobolev space$H^1(\Omega)$ an essential tool for solving complex mathematical problems.

Keywords : Spaces of distridutions $\mathcal{D}'(\Omega)$, Lebersgue spaces $L^2(\Omega)$, norm and scalar product, Hilbertian structure and Sobolev space $H^1(\Omega)$,

Introduction

Sobolev space $H^1(\Omega)$is a functional space modeled by Lebesgue space $L^2(\Omega)$. As a mathematical framework for functions with weak derivatives, it reveals its full power when given a Hilbertian structure. This Hilbertian structure gives Sobolev space $H^1(\Omega)$fundamental properties inherited from classical Hilbert spaces such as the definition of a scalar product and the completeness of the norm associated with it.

By studying the precise nature of this structure within Sobolev space $H^1(\Omega)$, this article will examine in detail the implications and advantages of this approach for the mathematical analysis of functions with weak derivatives.

Sobolev spaces $H^1(\Omega)$

1.Space$\mathcal{D}'(\Omega)$

1. **Definitions and properties**

Definition 1.1 . Let be Ωan open de $\mathbb{R}^n$and $\mathcal{D}(\Omega)$the space of infinitely differentiable functions with bounded support. We call distribution T on Ω, any linear and continuous form on $\mathcal{D}(\Omega)$.

(i)Linear T means: an application T of $\mathcal{D}(\Omega)$in $\mathbb{R}(ou\ \mathbb{C})$corresponding to a function $\varphi \in \mathcal{D}(\Omega)$, a number noted $\langle T, \varphi\rangle$such that: for all $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$and $\alpha, \beta \in \mathbb{C}$, we have:

$$\langle T, \alpha\varphi_1 + \beta\varphi_2\rangle = \alpha\langle T, \varphi_1\rangle + \beta\langle T, \varphi_2\rangle.$$

$(2i)$T continues means: if the sequence (φ_k)converges in $\mathcal{D}(\Omega)$towards φ, then the sequence $(\langle T, \varphi_k\rangle)$converges in the usual sense towards $\langle T, \varphi\rangle$.

We thus designate the space of distributions on $\Omega$by $\mathcal{D}'(\Omega)$, which is the topological dual of the space $\mathcal{D}(\Omega)$.

Proposition 1.2. The space $\mathcal{D}'(\Omega)$is a Banach space.

Evidence

$\mathcal{D}'(\Omega)$being a topological dual therefore $\mathcal{D}(\Omega)$, $\mathcal{D}(\Omega)$becomes a Banach space. ■

Definition 1.3. We call derivative T' of a distribution T , the linear and continuous form defined by:

$$\langle T', \varphi\rangle = -\langle T, \varphi'\rangle, \forall\varphi \in \mathcal{D}\ (\Omega)$$

In general, the order derivative $n$of the distribution T is defined by the relation:

$$\langle T^{(n)}, \varphi\rangle = (-1)^n\langle T, \varphi^{(n)}\rangle, \forall\varphi \in \mathcal{D}\ (\Omega).$$

Proposition 1.4. If $(T_k)$is a sequence of distributions converging $\mathcal{D}'(\Omega)$towards the distribution T then the sequence$\left(\frac{\partial T_k}{\partial x_i}\right)$ $converge\ dans\ \mathcal{D}'(\Omega)\ vers\ \frac{\partial T}{\partial x_i}$

Evidence :[6], page 66 ■

2. **Space$L^2(\Omega)$**
1. **Definitions and properties**

Definition 2.1. Let be Ωan open of $\mathbb{R}^n$.

We pose$L^2(\Omega) = \left\{f: \Omega \to \mathbb{R}: f\ intégrable\ et\ \int_\Omega |f(x)|^2 dx < +\infty\right\}$

Definition 2.2. Scalar product and norm

We provide $L^2(\Omega)$the scalar product $\langle .,. \rangle_{L^2(\Omega)}$and the norm $\|.\|_{L^2(\Omega)}$defined by:

$$\langle f,g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx, \forall (f,g) \in [L^2(\Omega)]^2$$

$$et \ \|f\|_{L^2(\Omega)} = \sqrt{\langle f,f\rangle_{L^2(\Omega)}} = \left(\int_\Omega f^2(x)dx\right)^{1/2}, \forall f \in L^2(\Omega)$$

Proposition 2.3. The space $\left(L^2(\Omega), \langle .,. \rangle_{L^2(\Omega)}\right)$is a Euclidean space.

Evidence. It is enough to show that $\langle .,. \rangle_{L^2(\Omega)}$is a dot product on $L^2(\Omega)$.

(i). Bilinearity

$\forall \ f,g,h \in L^2(\Omega) \ et \ \forall a,b \in \mathbb{R}$ we have :

$$\begin{aligned}\langle af+bg,h\rangle_{L^2(\Omega)} &= \int_\Omega [af(x)+bg(x)]h(x)dx \\ &= \int_\Omega [af(x)h(x)+bg(x)h(x)]dx \\ &= a\int_\Omega f(x)h(x)dx + b\int_\Omega g(x)h(x)dx \\ &= a\langle f,h\rangle_{L^2(\Omega)} + b\langle g,h\rangle_{L^2(\Omega)}\end{aligned}$$

And also:

$$\begin{aligned}\langle f,ag+bh\rangle_{L^2(\Omega)} &= \int_\Omega f(x)[ag(x)+bh(x)]dx \\ &= \int_\Omega [af(x)g(x)+bf(x)h(x)]dx \\ &= a\int_\Omega f(x)g(x)dx + b\int_\Omega f(x)h(x)dx \\ &= a\langle f,g\rangle_{L^2(\Omega)} + b\langle f,h\rangle_{L^2(\Omega)}\end{aligned}$$

This proves the bilinearity of$\langle .,. \rangle_{L^2(\Omega)}$

$(2i)$Symmetry of$\langle .,. \rangle_{L^2(\Omega)}$

$\forall \ f,g \in L^2(\Omega)$, we have :

$$\begin{aligned}\langle f,g\rangle_{L^2(\Omega)} &= \int_\Omega f(x)g(x)dx \\ &= \int_\Omega g(x)h(x)dx \\ &= \langle g,f\rangle_{L^2(\Omega)}\end{aligned}$$

So the symmetry is verified

$(3i)$Positivity defined

$\forall f \in L^2(\Omega)$

$Alors, \langle f, f \rangle_{L^2(\Omega)} = \int_\Omega [f(x)]^2 dx$

Since $[f(x)]^2$is always positive or zero, the integral

$\int_\Omega [f(x)]^2 dx \geq 0.$

$Donc, \langle f, f \rangle_{L^2(\Omega)} \geq 0.$

Moreover, if $\langle f, f \rangle_{L^2(\Omega)} = 0$,then$[f(x)]^2 = 0$ presque partout sur Ω.

This implies that $f(x) = 0$ almost everywhere on Ω.

So, $\langle f, f \rangle_{L^2(\Omega)} = 0 \Leftrightarrow f = 0$ presque partout sur Ω.

Therefore, the defined positivity is verified.

Under (i); $(2i)$ et $(3i)$, space $\left(L^2(\Omega), \langle .,. \rangle_{L^2(\Omega)}\right)$ has a Euclidean space structure.

■

Proposition 2.4. The space $\left(L^2(\Omega), \|.\|_{L^2(\Omega)}\right)$is a Banach space.

Evidence. Showing that $\left(L^2(\Omega), \|.\|_{L^2(\Omega)}\right)$is a Banach space amounts to verifying the following two axioms:

(i) $L^2(\Omega)$ is a complete vector space

$(2i)$The standard $\|.\|_{L^2(\Omega)}$on $L^2(\Omega)$is complete.

(i) $L^2(\Omega)$ is a complete vector space.

We show that each Cauchy sequence in $L^2(\Omega)$converges to an element of$L^2(\Omega)$.

Consider (f_n)a Cauchy sequence in $L^2(\Omega)$. It means that

$\forall \varepsilon > 0, \exists k \in \mathbb{N}$such as$\forall m, n \geq k,\ \|f_m - f_n\|_{L^2(\Omega)} \leq \varepsilon$

As $(f_n)$is a Cauchy sequence, it is uniformly convergent almost everywhere to a function f.This implies that the sequence $(f_n(x))$converges almost everywhere to $f(x)$when $n$tends to$+\infty$.

Let us show that $f$is in $L^2(\Omega)$. Since $\|f_n - f\|_{L^2(\Omega)} \longrightarrow 0$, lorsque n tend vers $+\infty$,then (f_n)also converges $f$to norm$L^2(\Omega)$,

$$C'est - \text{à} - dire\ que\ \lim_{n \to +\infty} \int_\Omega |f_n(x) - f(x)|^2 dx = 0.$$

Therefore, f is in $L^2(\Omega)$, and therefore, $L^2(\Omega)$ is a complete vector space.

$(2i)$The standard $\|.\|_{L^2(\Omega)}$ sur $L^2(\Omega)$ is complete.

It suffices to show that any Cauchy sequence (f_n) in $L^2(\Omega)$converges in $L^2(\Omega)$under this norm.

As we have already established, any Cauchy sequence (f_n) in $L^2(\Omega)$ converges uniformly almost everywhere on $\Omega$to a function f,and also converges on f to norm $L^2(\Omega)$.

Thus, the standard $\|.\|_{L^2(\Omega)}$on $L^2(\Omega)$ is complete.

In conclusion, $\left(L^2(\Omega), \|.\|_{L^2(\Omega)}\right)$ is a Banach space. ■

3. Space$\boldsymbol{H^1(\Omega)}$

Definitions and properties

Definition 3.1. Given Ωan open de $\mathbb{R}^2$, we define the Sobolev space $H^1(\Omega)$ by:

$$H^1(\Omega) = \left\{ f \in L^2(\Omega) : \frac{\partial f}{\partial x_i} \in L^2(\Omega), \forall i = 1,2 \right\}$$

with $\frac{\partial f}{\partial x_i}$ partial derivatives of f taken in the sense of distributions defined by:

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i} dx_1 dx_2 = -\int_\Omega \frac{\partial f}{\partial x_i} \varphi dx_1 dx_2, \forall \varphi \in \mathcal{D}(\Omega)$$

Definition 3.2. The scalar product $\langle .,. \rangle$ and the norm $\|.\|$ on $H^1(\Omega)$are defined by:

$$\langle f, g \rangle_{H^1(\Omega)} = \int_\Omega \left(fg + \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_2} \right) dx_1 dx_2$$

$$\forall (f, g) \in [H^1(\Omega)]^2$$

$$et \; \|f\|_{H^1(\Omega)} = \left(\|f\|^2_{L^2(\Omega)} + \left\| \frac{\partial f}{\partial x_1} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial f}{\partial x_2} \right\|^2_{L^2(\Omega)} \right)^{1/2}$$

Proposition 3.3. The space $\left(H^1(\Omega), \langle .,. \rangle_{H^1(\Omega)}\right)$is a Euclidean space.

Evidence. The space $\left(H^1(\Omega), \langle .,. \rangle_{H^1(\Omega)}\right)$is a Euclidean space if $\langle .,. \rangle_{H^1(\Omega)}$is a dot product on $H^1(\Omega)$x $H^1(\Omega)$.A dot product is a positive definite symmetric bilinear form.

(i) Bilinearity of$\langle .,. \rangle_{H^1(\Omega)}$

$$\forall f_1, f_2, g_1, g_2 \in H^1(\Omega), \forall a, b, c, d \in \mathbb{R} : \langle af_1 + bf_2, cg_1 + dg_2 \rangle_{H^1(\Omega)}$$

$$= ac\langle f_1, g_1 \rangle + ad\langle f_1, g_2 \rangle + bc\langle f_2, g_1 \rangle + bd\langle f_2, g_2 \rangle$$

$$= \int_{\Omega} \Big[(af_1 + bf_2)(cg_1 + dg_2) + \frac{\partial}{\partial x_1}(af_1 + bf_2)\frac{\partial}{\partial x_1}(ag_1 + dg_2)$$
$$+ \frac{\partial}{\partial x_2}(af_1 + bf_2)\frac{\partial}{\partial x_2}(cg_1 + dg_2)\Big] dx_1 dx_{2.}$$

$$= \int_{\Omega} \Big[af_1.cg_1 + af_1.dg_2 + bf_2.cg_1 + bf_2.dg_2 + \left(a\frac{\partial f_1}{\partial x_1} + b\frac{\partial f_2}{\partial x_2}\right)\left(c\frac{\partial g_1}{\partial x_1} + d\frac{\partial g_2}{\partial x_1}\right)$$
$$+ \left(a\frac{\partial f_1}{\partial x_2} + b\frac{\partial f_2}{\partial x_2}\right)\left(c\frac{\partial g_1}{\partial x_2} + d\frac{\partial g_2}{\partial x_2}\right)\Big] dx_1 dx_{2.}$$
$$= \int_{\Omega} \Big[acf_1g_1 + adf_1g_2 + bcf_2g_1 + bdf_2g_2 + ac\frac{\partial f_1}{\partial x_1}\frac{\partial g_1}{a x_1} + ad\frac{\partial f_1}{\partial x_1}\frac{\partial g_2}{\partial x_1} + bc\ \frac{\partial f_2}{\partial x_1}\frac{\partial g_1}{\partial x_1}$$
$$+ bd\ \frac{\partial f_2}{\partial x_1}.\frac{\partial g_2}{\partial x_1} + ac\frac{\partial f_1}{\partial x_2}\frac{\partial g_1}{\partial x_2} + ad\frac{\partial f_1}{\partial x_2}\frac{\partial g_2}{\partial x_2} + bc\ \frac{\partial f_2}{\partial x_2}\frac{\partial g_1}{\partial x_2}$$
$$+ bd\ \frac{\partial f_2}{\partial x_2}.\frac{\partial g_2}{\partial x_2}\Big] dx_1 dx_{2.}$$

$$= \int_{\Omega} \left(ac\ f_1g_1 + ac\frac{\partial f_1}{\partial x_1}\frac{\partial g_1}{\partial x_1} + ac\ \frac{\partial f_1}{\partial x_2}\frac{\partial g_1}{\partial x_2}\right) dx_1 dx_2$$
$$+ \int_{\Omega} \left(ad\ f_1g_2 + ad\frac{\partial f_1}{\partial x_1}\frac{\partial g_2}{\partial x_1} + ad\ \frac{\partial f_1}{\partial x_2}\frac{\partial g_2}{\partial x_2}\right) dx_1 dx_2$$
$$+ \int_{\Omega} \left(bc\ f_2g_1 + bc\frac{\partial f_2}{\partial x_1}\frac{\partial g_1}{\partial x_1} + bc\ \frac{\partial f_2}{\partial x_2}\frac{\partial g_1}{\partial x_2}\right) dx_1 dx_2$$
$$+ \int_{\Omega} \left(bd\ f_2g_2 + bd\frac{\partial f_2}{\partial x_1}\frac{\partial g_2}{\partial x_1} + bd\ \frac{\partial f_2}{\partial x_2}\frac{\partial g_2}{\partial x_2}\right) dx_1 dx_2$$

$$= ac \int_{\Omega} \left(f_1g_1 + \frac{\partial f_1}{\partial x_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f_1}{\partial x_2}\frac{\partial g_1}{\partial x_2}\right) dx_1 dx_2 + ad \int_{\Omega} \left(f_1g_2 + \frac{\partial f_1}{\partial x_1}\frac{\partial g_2}{\partial x_1} + \frac{\partial f_1}{\partial x_2}\frac{\partial g_2}{\partial x_2}\right) dx_1 dx_2$$

$$+ad \int_{\Omega} \left(f_1g_2 + \frac{\partial f_1}{\partial x_1}\frac{\partial g_2}{\partial x_1} + \frac{\partial f_1}{\partial x_2}\frac{\partial g_2}{\partial x_2}\right) dx_1 dx_2$$

$$+bc \int_{\Omega} \left(f_2g_1 + \frac{\partial f_2}{\partial x_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\frac{\partial g_1}{\partial x_2}\right) dx_1 dx_2$$

$$+bd \int_{\Omega} \left(f_2g_2 + \frac{\partial f_2}{\partial x_1}\frac{\partial g_2}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\frac{\partial g_2}{\partial x_2}\right) dx_1 dx_2$$

$$= ac\langle f_1, g_1\rangle + ad\langle f_1, g_2\rangle + bc\langle f_2, g_1\rangle + bd\langle f_2, g_2\rangle$$

Bilinearity is verified.

$(2i)$Symmetry of$\langle .,. \rangle_{H^1(\Omega)}$

In fact, $\forall f, g \in\ H^1\ (\Omega)$we have:

$$\langle f, g\rangle_{H^1(\Omega)} \quad = \int_{\Omega} \left(fg + \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_2}\right) dx_1 dx_2$$

$$= \int_\Omega \left(gf + \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_2}\frac{\partial f}{\partial x_2} \right) dx_1 dx_2$$

$$= \langle g, f \rangle_{H^1(\Omega)}$$

This ensures the symmetry of$\langle .,. \rangle_{H^1(\Omega)}$

$(3i)$Positivity defined

$\forall f \in H^1(\Omega)$we have :

$$\langle f, f \rangle_{H^1(\Omega)} = \int_\Omega \left[f^2 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \right] dx_1 dx_2$$

$$= \|f\|^2_{H^1(\Omega)} > 0$$

Which ensures the positivity of$\langle .,. \rangle_{H^1(\Omega)}$

$\langle f, g \rangle_{H^1(\Omega)}$is well defined on$H^1(\Omega)$

$$Car\ \langle f, f \rangle_{H^1(\Omega)} = 0 \Leftrightarrow f = 0\ presque\ partout\ sur\ \Omega.$$

$$On\ a:\ \langle f, f \rangle_{H^1(\Omega)} = 0 \Leftrightarrow \int_\Omega \left[f^2 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \right] dx_1 dx_2 = 0$$

The integral of a positive function is zero when the integrand is zero almost everywhere on Ω. This means that $f = 0\ p.p\ sur\ \Omega, \frac{\partial f}{\partial x_1} = 0.p.p\ sur\ \Omega$and

$\frac{\partial f}{\partial x_2} = 0\ p.p\ sur\ \Omega.$ which implies that $f = 0\ p.p$onΩ.

We have shown that $\langle .,. \rangle$is a scalar product on $H^1(\Omega)$, therefore $\left(H^1(\Omega), \langle .,. \rangle_{H^1(\Omega)}\right)$is a Euclidean space. ■

Proposition 3.4. The space $\left(H^1(\Omega), \|.\|_{H^1(\Omega)}\right)$is a standardized space.

Evidence. $H^1(\Omega)$is a norm space if $\|.\|_{H^1(\Omega)}$is a norm on $H^1(\Omega)$.

$$\|f\|_{H^1(\Omega)} = \left(\|f\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_2}\right\|^2_{L^2(\Omega)} \right)^{1/2}$$

- **Separation**

$\|f\|_{H^1(\Omega)} = 0 \Leftrightarrow f = 0\ p.p.\Omega$

$$\|f\|_{H^1(\Omega)} = 0 \Leftrightarrow \left(\|f\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_2}\right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} = 0$$

The square root of a positive radicand is zero when the radicand is zero. SO

$\|f\|_{L^2(\Omega)} = 0 \Leftrightarrow f = 0\ p.p\ on\ \Omega$

$$and \begin{cases} \left\|\frac{\partial f}{\partial x_1}\right\|^2_{L^2(\Omega)} = 0 \\ \left\|\frac{\partial f}{\partial x_2}\right\|^2_{L^2(\Omega)} = 0 \end{cases} \Leftrightarrow f = 0 \ \ p.p \ on \ \Omega$$

SO$f = 0 \, p.p \, sur \, \Omega$

- **Homogeneity**

$\|\lambda f\|_{H^1(\Omega)} = |\lambda| \|f\|_{H^1(\Omega)}, \forall \lambda \in \mathbb{R}.$

$$\|\lambda f\|_{H^1(\Omega)} = \left(\|\lambda f\|^2_{L^2(\Omega)} + \left\|\frac{\partial(\lambda f)}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial(\lambda f)}{\partial x_2}\right\|^2_{L^2(\Omega)} \right)^{1/2}$$

$$= \left(\lambda^2 \|f\|^2_{L^2(\Omega)} + \lambda^2 \left\|\frac{\partial f}{\partial x_1}\right\|^2_{L^2(\Omega)} + \lambda^2 \left\|\frac{\partial f}{\partial x_2}\right\|^2_{L^2(\Omega)} \right)^{1/2}$$

$$= |\lambda| \left(\|f\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_2}\right\|^2_{L^2(\Omega)} \right)^{1/2}$$

$$= |\lambda| \|f\|_{H^1(\Omega)}$$

- **Triangle inequality**

$\|f + g\|_{H^1(\Omega)} \le \|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)}, \forall (f, g) \in [H^1(\Omega)]^2$

$$\|f + g\|_{H^1(\Omega)} = \left(\|f + g\|^2_{L^2(\Omega)} + \left\|\frac{\partial(f+g)}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial(f+g)}{\partial x_2}\right\|^2_{L^2(\Omega)} \right)^{1/2}$$

$$= \left(\|f + g\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_2} + \frac{\partial g}{\partial x_2}\right\|^2_{L^2(\Omega)} \right)^{1/2}$$

$$\le \left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial g}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_2}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial g}{\partial x_2}\right\|^2_{L^2(\Omega)} \right)^{1/2}$$

Thanks to Minkowski's inequality, we obtain

$$\le \left(\|f\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial x_2}\right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} + \left(\|g\|^2_{L^2(\Omega)} + \left\|\frac{\partial g}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial g}{\partial x_2}\right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

$$\le \|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)}$$

$\|.\|_{H^1(\Omega)}$is indeed a standard on$H^1(\Omega)$

Hence $\left(H^1(\Omega), \|.\|_{H^1(\Omega)}\right)$is a standardized space. ∎

Proposition 3.5. Space H^1 (Ω)has a Hilbertian structure.

Evidence. It suffices to show that H^1 $(\Omega)$is a Hilbert space for the norm$\|.\|_{H^1(\Omega)}$.

Indeed,

(i) $\langle .,.\rangle_{H^1(\Omega)}$is a scalar product on H^1 (Ω)x H^1 (Ω)according to proposition (3.3)

$\|.\|_{H^1(\Omega)}$is a norm on H^1 (Ω)according to proposition (3.4)

$(2i)$Let us show that H^1 $(\Omega)$is complete for the norm$\|.\|_{H^1(\Omega)}$

Consider $(f_k)\in H^1$ (Ω)a Cauchy sequence for the norm $\|.\|_{H^1(\Omega)}$. That's to say

$$\forall\varepsilon>0, \exists\ k\in\mathbb{N}>0/n,m\geq k\colon \|f_n-f_m\|_{H^1(\Omega)}\leq\varepsilon^2$$

$$\|f_n-f_m\|^2_{L^2(\Omega)}+\left\|\frac{\partial f_n}{\partial x_1}-\frac{\partial f_m}{\partial x_1}\right\|^2_{L^2(\Omega)}+\left\|\frac{\partial f_n}{\partial x_2}-\frac{\partial f_m}{\partial x_2}\right\|^2_{L^2(\Omega)}\leq\varepsilon^2$$

That implies that :

$\|f_n-f_m\|^2_{L^2(\Omega)}\leq\varepsilon^2, \left\|\frac{\partial f_n}{\partial x_i}-\frac{\partial f_m}{\partial x_i}\right\|^2_{L^2(\Omega)}\leq\varepsilon^2$ For$i=1,2$

we have: $(f_k)$is from Cauchy for$\|.\|_{L^2(\Omega)}$

$\left(\frac{\partial f_k}{\partial x_i}\right)$is also from Cauchy for$\|.\|_{L^2(\Omega)}$

$(i=1,2)$

Gold L^2 $(\Omega)$is complete for $\|.\|_{L^2(\Omega)}$, this implies that:

$\exists f\in L^2$ (Ω)such as(f_k) $converge\ to\ f\ on\ L^2\ (\Omega)\ relatively\ to\ \|.\|_{L^2(\Omega)}$

$$\Leftrightarrow\lim_{n\to\infty}\|f_k-f\|_{L^2(\Omega)}=0$$

$\exists g_i\in L^2$ (Ω)such as$\left(\frac{\partial f_k}{\partial x_i}\right)$ $converge\ to\ g_i\ on\ L^2\ (\Omega)\ rrelatively\ to\ \|.\|_{L^2(\Omega)}$

$$\Leftrightarrow\lim_{n\to\infty}\left\|\frac{\partial f_k}{\partial x_i}-g_i\right\|_{L^2(\Omega)}=0$$

It remains to show that the sequence (f_k) converges to the function $f\in H^1$ $(\Omega)$with

$$\lim_{n\to\infty}\|f_k-f\|_{H^1(\Omega)}=0$$

Gold$(f_k)\in H^1$ $(\Omega)\Leftrightarrow\begin{cases}(f_k)\in L^2\ (\Omega)\\ \left(\frac{\partial f_k}{\partial x_i}\right)\in L^2\ (\Omega)\ for\ i=1,2\end{cases}$

$$avec \int_\Omega f_k \frac{\partial \varphi}{\partial x_i} = -\int_\Omega \frac{\partial f_k}{\partial x_i}\varphi, \forall \varphi \in \mathcal{D}(\Omega), \forall k \in \mathbb{N}$$

$$\left|\int_\Omega f_k \frac{\partial \varphi}{\partial x_i} - \int_\Omega f \frac{\partial \varphi}{\partial x_i}\right| = \left|\int_\Omega (f_k - f)\frac{\partial \varphi}{\partial x_i}\right|$$

By increasing the left side by the Cauchy-Schwarz inequality, we obtain

$$\left|\int_\Omega f_k \frac{\partial \varphi}{\partial x_i} - \int_\Omega f \frac{\partial \varphi}{\partial x_i}\right| = \left|\int_\Omega (f_k - f)\frac{\partial \varphi}{\partial x_i}\right|$$

$$\leq \|f_k - f\|_{L^2(\Omega)} . \left\|\frac{\partial \varphi}{\partial x_i}\right\|_{L^2(\Omega)}$$

$\|f_k - f\|_{L^2(\Omega)} \to 0$by completeness of$L^2(\Omega)$.

We obtain $\lim_{k\to\infty} \int_\Omega f_k \frac{\partial \varphi}{\partial x_i} = \int_\Omega f \frac{\partial \varphi}{\partial x_i}$

Likewise for the right hand side we obtain

$$\lim_{k\to\infty} \int_\Omega \frac{\partial f_k}{\partial x_i}\varphi = \int_\Omega g_i \varphi$$

By passing to the limit, we obtain:

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi, \forall \varphi \in \mathcal{D}(\Omega), (i = 1,2)$$

This implies that $g_i = \frac{\partial f}{\partial x_i}$derivative in the sense of distributions,$f \in H^1(\Omega)$.

We have: (f_k)converges towards $f$in $L^2(\Omega)$and$\left(\frac{\partial f_k}{\partial x_i}\right)$ $converge\ vers\ g_i = \frac{\partial f}{\partial x_i}\ dans$ $L^2(\Omega)$

Moreover :

$$\|f_k - f\|_{H^1(\Omega)}^2 = \|f_k - f\|_{L^2(\Omega)}^2 + \left\|\frac{\partial f_k}{\partial x_1} - \frac{\partial f}{\partial x_1}\right\|_{L^2(\Omega)}^2 + \left\|\frac{\partial f_k}{\partial x_2} - \frac{\partial f}{\partial x_2}\right\|_{L^2(\Omega)}^2$$

$\|f_k - f\|_{L^2(\Omega)}^2 \to 0$ by completeness of$L^2(\Omega)$

$\left\|\frac{\partial f_k}{\partial x_1} - \frac{\partial f}{\partial x_2}\right\|_{L^2(\Omega)}^2 \longrightarrow 0$ by completeness of$L^2(\Omega)$

$\left\|\frac{\partial f_k}{\partial x_2} - \frac{\partial f}{\partial x_2}\right\|_{L^2(\Omega)}^2 \longrightarrow 0$ by completeness of$L^2(\Omega)$

This implies that the sequence (f_k) converges to the function f in $H^1(\Omega)$.

We then write$(f_k) \longmapsto f \Leftrightarrow \lim_{k\to\infty} \|f_k - f\|_{H^1(\Omega)} = 0$

Which completes the proof.

Therefore $H^1(\Omega)$has a Hilbertian structure. ■

Conclusion

Sobolev space $H^1(\Omega)$, we first presented an overview of distributions and Lebesgue spaces $L^2(\Omega)$. Secondly, we established that the Sobolev space $H^1(\Omega)$has a remarkable Hilbertian structure.

By constructing an appropriate scalar product involving the functions and their gradients, we showed that this space is equipped with a norm and a scalar product satisfying the essential axioms of bilinearity, positivity, symmetry and the Cauchy-Schwarz inequality. to define a Hilbert space. The completeness of $H^1(\Omega)$guarantees that any Cauchy sequence in this space converges to a limit in $H^1(\Omega)$, thus confirming its nature as a Hilbert space.

References

[1]Lacroix-Sonner MT, 1997, Distributions, Sobolev spaces, applications, Ellipses, Paris.

[2]Bhattacharyya PK, 2012, Distributions-generalized functions with applications in sobolev spaces. From Gruyter , Boston.

[3]Daniel Li, 2013; functional analysis course with 200 corrected exercises. Ellipses, Paris

[4]Sobolev spaces , April 29.

[5] Haim Brezis : 1983; Functional analysis course theory and applications. Masson, Paris.

[6]Claire David PG, 2015; Partial differential equations, courses and corrected exercises. Dunod , Paris….