THE PARADIGM OF COMPLEX PROBABILITY AND MONTÉ CARLO METHODS

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CHAPTER ONE

THE PARADIGM OF COMPLEX PROBABILITY APPLIED TO MONTÉ CARLO METHODS

"Thus, joining the rigor of the demonstrations of science to the uncertainty of fate, and reconciling these two seemingly contradictory things, it can, taking its name from both, appropriately arrogate to itself this astonishing title: the geometry of chance."

Blaise Pascal.

"You believe in the God who plays dice, and I in complete law and order."

Albert Einstein, Letter to Max Born.

Abstract: In 1933, Andrey Nikolaevich Kolmogorov established the system of five axioms that define the concept of mathematical probability. This system can be developed to include the set of imaginary numbers and this by adding a supplementary three original axioms. Therefore, any experiment can be performed in the set $\mathcal{C}$ of complex probabilities which is the summation of the set $\mathcal{R}$ of real probabilities and the set $\mathcal{M}$ of imaginary probabilities. The purpose here is to include additional imaginary dimensions to the experiment taking place in the "real" laboratory in $\mathcal{R}$ and hence to evaluate all the probabilities. Consequently, the probability in the entire set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is permanently equal to one no matter what the stochastic distribution of the input random variable in $\mathcal{R}$ is, therefore the outcome of the probabilistic experiment in $\mathcal{C}$ can be determined perfectly. This is due to the fact that the probability in $\mathcal{C}$ is calculated after subtracting from the degree of our knowledge the chaotic factor of the random experiment. This novel complex probability paradigm will be applied to the classical probabilistic Monte Carlo numerical methods and to prove as well the convergence of these stochastic procedures in an original way.

Keywords: complex set, degree of our knowledge, chaotic factor, complex random vector, probability norm, simulation, convergence probability, divergence probability.

NOMENCLATURE

$\mathcal{R}$	= real set of events
$\mathcal{M}$	= imaginary set of events
$\mathcal{C}$	= complex set of events
i	= the imaginary number where $i = \sqrt{-1}$ or $i^2 = -1$
EKA	= Extended Kolmogorov's Axioms
CPP	= Complex Probability Paradigm
P_{rob}	= probability of any event
P_r	= probability in the real set $\mathcal{R}$ = probability of convergence in $\mathcal{R}$
P_m	= probability in the imaginary set $\mathcal{M}$ corresponding to the real probability in $\mathcal{R}$ = probability of divergence in $\mathcal{M}$

Pc = probability of an event in $\mathcal{R}$ with its associated event in $\mathcal{M}$ = probability in the complex probability set $\mathcal{C}$

R_E = the exact result of the random experiment

R_A = the approximate result of the random experiment

z = complex probability number = sum of P_r and P_m = complex random vector

DOK = $|z|^2$ = the degree of our knowledge of the random system or experiment, it is the square of the norm of z

Chf = the chaotic factor of z

$MChf$ = magnitude of the chaotic factor of z

N = number of random vectors = number of iterations cycles

N_C = number of random vectors = number of iterations cycles till the convergence of Monte Carlo method to R_E

Z = the resultant complex random vector = $\sum_{j=1}^{N} z_j$

$DOK_Z = \frac{|Z|^2}{N^2}$ = the degree of our knowledge of the whole stochastic system

$Chf_Z = \frac{Chf}{N^2}$ = the chaotic factor of the whole stochastic system

$MChf_Z$ = magnitude of the chaotic factor of the whole stochastic system

Z_U = the resultant complex random vector corresponding to a uniform random distribution

DOK_{Z_U} = the degree of our knowledge of the whole stochastic system corresponding to a uniform random distribution

Chf_{Z_U} = the chaotic factor of the whole stochastic system corresponding to a uniform random distribution

$MChf_{Z_U}$ = the magnitude of the chaotic factor of the whole stochastic system corresponding to a uniform random distribution

Pc_U = probability in the complex probability set $\mathcal{C}$ of the whole stochastic system corresponding to a uniform random distribution

I- Introduction

Firstly, in this introductory section an overview of Monte Carlo methods will be done. Before the Monte Carlo method was developed, simulations tested a previously understood deterministic problem, and statistical sampling was used to estimate uncertainties in the simulations. Monte Carlo simulations invert this approach, solving deterministic problems using a probabilistic analog (once can refer to Simulated annealing).

An early variant of the Monte Carlo method can be seen in the Buffon's needle experiment, in which π can be estimated by dropping needles on a floor made of parallel and equidistant strips. In the 1930s, Enrico Fermi first experimented with the Monte Carlo method while studying neutron diffusion, but did not publish anything on it. [1]

The modern version of the Markov Chain Monte Carlo method was invented in the late 1940s by Stanislaw Ulam, while he was working on nuclear weapons projects at the Los Alamos National Laboratory. Immediately after Ulam's breakthrough, John von Neumann understood its importance and programmed the ENIAC computer to carry out Monte Carlo calculations. In 1946, physicists at Los Alamos Scientific Laboratory were investigating radiation shielding and the distance that neutrons would likely travel through various materials. Despite having most of the necessary data, such as the average distance a neutron would travel in a substance before it collided with an atomic nucleus, and how much energy the neutron was likely to give off following a collision, the Los Alamos physicists were unable to solve the problem using conventional, deterministic mathematical methods. Ulam had the idea of using random experiments. He recounts his inspiration as follows:

“The first thoughts and attempts I made to practice [the Monte Carlo Method] were suggested by a question which occurred to me in 1946 as I was convalescing from an illness and playing solitaires. The question was what are the chances that a Canfield solitaire laid out with 52 cards will come out successfully? After spending a lot of time trying to estimate them by pure combinatorial calculations, I wondered whether a more practical method than "abstract thinking" might not be to lay it out say one hundred times and simply observe and count the number of successful plays. This was already possible to envisage with the beginning of the new era of fast computers, and I immediately thought of problems of neutron diffusion and other questions of mathematical physics, and more generally how to change processes described by certain differential equations into an equivalent form interpretable as a succession of random operations. Later [in 1946], I described the idea to John von Neumann, and we began to plan actual calculations.” [2]

Being secret, the work of von Neumann and Ulam required a code name. [3] A colleague of von Neumann and Ulam, Nicholas Metropolis, suggested using the name Monte Carlo, which refers to the Monte Carlo Casino in Monaco where Ulam's uncle would borrow money from relatives to gamble. [1] Using lists of "truly random" random numbers was extremely slow, but von Neumann developed a way to calculate pseudorandom numbers, using the middle-square method. Though this method has been criticized as crude, von Neumann was aware of this: he justified it as being faster than any other method at his disposal, and also noted that when it went awry it did so obviously, unlike methods that could be subtly incorrect. [4]

Monte Carlo methods were central to the simulations required for the Manhattan Project, though severely limited by the computational tools at the time. In the 1950s they were used at Los Alamos for early work relating to the development of the hydrogen bomb, and became popularized in the fields of physics, physical chemistry, and operations research. The Rand Corporation and the U.S. Air Force were two of the major organizations responsible for funding and disseminating information on Monte Carlo methods during this time, and they began to find a wide application in many different fields.

The theory of more sophisticated mean field type particle Monte Carlo methods had certainly started by the mid-1960s, with the work of Henry P. McKean Jr. on Markov interpretations of a class of nonlinear parabolic partial differential equations arising in fluid mechanics. [5,6] We also quote an earlier pioneering article by Theodore E. Harris and Herman Kahn, published in 1951,

using mean field genetic-type Monte Carlo methods for estimating particle transmission energies. [7] Mean field genetic type Monte Carlo methodologies are also used as heuristic natural search algorithms (a.k.a. Metaheuristic) in evolutionary computing. The origins of these mean field computational techniques can be traced to 1950 and 1954 with the work of Alan Turing on genetic type mutation-selection learning machines [8] and the articles by Nils Aall Barricelli at the Institute for Advanced Study in Princeton, New Jersey. [9,10]

Quantum Monte Carlo, and more specifically Diffusion Monte Carlo methods can also be interpreted as a mean field particle Monte Carlo approximation of Feynman-Kac path integrals. [11-17] The origins of Quantum Monte Carlo methods are often attributed to Enrico Fermi and Robert Richtmyer who developed in 1948 a mean field particle interpretation of neutron-chain reactions, [18] but the first heuristic-like and genetic type particle algorithm (a.k.a. Resampled or Reconfiguration Monte Carlo methods) for estimating ground state energies of quantum systems (in reduced matrix models) is due to Jack H. Hetherington in 1984 [17] In molecular chemistry, the use of genetic heuristic-like particle methodologies (a.k.a. pruning and enrichment strategies) can be traced back to 1955 with the seminal work of Marshall. N. Rosenbluth and Arianna. W. Rosenbluth. [19]

The use of Sequential Monte Carlo in advanced signal processing and Bayesian inference is more recent. It was in 1993, that Gordon et al., published in their seminal work [20] the first application of a Monte Carlo resampling algorithm in Bayesian statistical inference. The authors named their algorithm 'the bootstrap filter', and demonstrated that compared to other filtering methods, their bootstrap algorithm does not require any assumption about that state-space or the noise of the system. We also quote another pioneering article in this field of Genshiro Kitagawa on a related "Monte Carlo filter", [21] and the ones by Pierre Del Moral [22] and Himilcon Carvalho, Pierre Del Moral, André Monin and Gérard Salut [23] on particle filters published in the mid-1990s. Particle filters were also developed in signal processing in the early 1989-1992 by P. Del Moral, J.C. Noyer, G. Rigal, and G. Salut in the LAAS-CNRS in a series of restricted and classified research reports with STCAN (Service Technique des Constructions et Armes Navales), the IT company DIGILOG, and the LAAS-CNRS (the Laboratory for Analysis and Architecture of Systems) on RADAR/SONAR and GPS signal processing problems. [24-29] These Sequential Monte Carlo methodologies can be interpreted as an acceptance-rejection sampler equipped with an interacting recycling mechanism.

From 1950 to 1996, all the publications on Sequential Monte Carlo methodologies including the pruning and resample Monte Carlo methods introduced in computational physics and molecular chemistry, present natural and heuristic-like algorithms applied to different situations without a single proof of their consistency, nor a discussion on the bias of the estimates and on genealogical and ancestral tree-based algorithms. The mathematical foundations and the first rigorous analysis of these particle algorithms are due to Pierre Del Moral [22,30] in 1996. Branching type particle methodologies with varying population sizes were also developed in the end of the 1990s by Dan Crisan, Jessica Gaines and Terry Lyons, [31-33] and by Dan Crisan, Pierre Del Moral and Terry Lyons. [34] Further developments in this field were developed in 2000 by P. Del Moral, A. Guionnet and L. Miclo. [12,35,36]

Finally, and to conclude, this research work is organized as follows: After the introduction in section I, the purpose and the advantages of the present work are presented in section II. Afterward, in section III, we will explain and illustrate the complex probability paradigm with its original parameters and interpretation. In section IV, the Monte Carlo techniques of integration and simulation will be explained. In section V, I will extend Monte Carlo methods to the imaginary and complex probability sets and hence link this concept to my novel complex probability paradigm. Moreover, in section VI, I will prove the convergence of Monte Carlo methods using the concept of the resultant complex random vector *Z*. Furthermore, in section VII we will evaluate the original paradigm parameters and in section VIII a flowchart of the complex probability and Monte Carlo methods prognostic model will be drawn. Additionally, in section IX simulations of Monte Carlo methods will be accomplished in the continuous and discrete cases. Finally, I conclude the work by doing a comprehensive summary in section X, and then present the list of references cited in the current research work.

II- The Purpose and the Advantages of the Present Work [37-90]

In this section we will present the purpose and the advantages of the current research work. Computing probabilities is the main work of classical probability theory. Adding new dimensions to the stochastic experiments will lead to a deterministic expression of probability theory. This is the original idea at the foundations of this work. Actually, the theory of probability is a nondeterministic system in its essence; that means that the events outcomes are due to chance and randomness. The addition of novel imaginary dimensions to the chaotic experiment occurring in the set $\mathcal{R}$ will yield a deterministic experiment and hence a stochastic event will have a certain result in the complex probability set $\mathcal{C}$. If the random event becomes completely predictable then we will be fully knowledgeable to predict the outcome of stochastic experiments that arise in the real world in all stochastic processes. Consequently, the work that has been accomplished here was to extend the real probabilities set $\mathcal{R}$ to the deterministic complex probabilities set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ by including the contributions of the set $\mathcal{M}$ which is the imaginary set of probabilities. Therefore, since this extension was found to be successful, then a novel paradigm of stochastic sciences and prognostic was laid down in which all stochastic phenomena in $\mathcal{R}$ was expressed deterministically. I called this original model "the Complex Probability Paradigm" that was initiated and illustrated in my previous research publications.

Accordingly, the advantages and the purpose of the current paper are to:

1- Extend classical probability theory to the set of complex numbers, therefore to link the theory of probability to the field of complex variables and analysis. This job was started and elaborated in my previous works.

2- Apply the new axioms of probability and paradigm to Monte Carlo methods.

3- Show that all stochastic phenomena can be expressed deterministically in the set of complex probabilities $\mathcal{C}$.

4- Measure and compute both the degree of our knowledge and the chaotic factor of Monte Carlo methods.

5- Draw and illustrate the graphs of the parameters and functions of the original paradigm corresponding to Monte Carlo methods.

6- Show that the classical concept of probability is always equal to one in the complex set; hence, no randomness, no chaos, no uncertainty, no ignorance, no disorder, and no unpredictability exist in:

$$\mathcal{C}\text{ (complex set)} = \mathcal{R}\text{ (real set)} + \mathcal{M}\text{ (imaginary set)}.$$

7- Prove the convergence of the stochastic Monte Carlo procedures in an original way by using the newly defined axioms and paradigm.

8- Pave the way to implement this novel model to other areas in stochastic processes and to the field of prognostics. These will be the topics of my future research works.

Concerning some applications of the original elaborated paradigm and as a future work, it can be applied to any random phenomena using Monte Carlo methods whether in the discrete or in the continuous cases.

Furthermore, compared with existing literature, the main contribution of the present research work is to apply the novel paradigm of complex probability to the concepts and techniques of the stochastic Monte Carlo methods and simulations.

The following figure shows the main purposes of the Complex Probability Paradigm (*CPP*) (Figure 1).

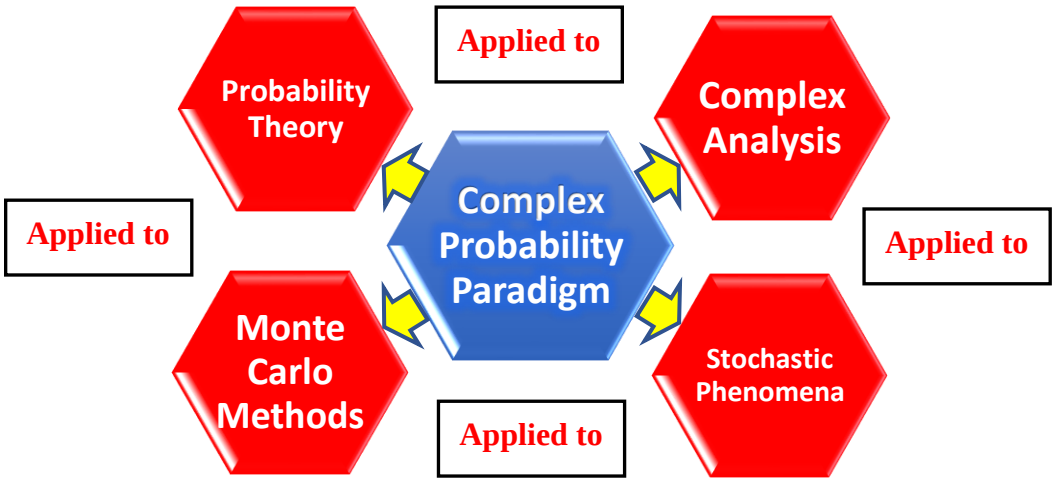


Figure 1. The diagram of the main purposes of the Complex Probability Paradigm

III- The Complex Probability Paradigm

III-1- The Original Andrey Nikolaevich Kolmogorov System of Axioms

The simplicity of Kolmogorov's system of axioms may be surprising. Let E be a collection of elements $\{E_1, E_2, \ldots\}$ called elementary events and let F be a set of subsets of E called random events. The five axioms for a finite set E are [91-96]:

Axiom 1: F is a field of sets.

Axiom 2: F contains the set E.

Axiom 3: A non-negative real number $P_{rob}(A)$, called the probability of A, is assigned to each set A in F. We have always $0 \le P_{rob}(A) \le 1$.

Axiom 4: $P_{rob}(E)$ equals 1.

Axiom 5: If A and B have no elements in common, the number assigned to their union is:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B)$$

hence, we say that A and B are disjoint; otherwise, we have:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B) - P_{rob}(A \cap B)$$

And we say also that: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B/A) = P_{rob}(B) \times P_{rob}(A/B)$ which is the conditional probability. If both A and B are independent then: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B)$.

Moreover, we can generalize and say that for N disjoint (mutually exclusive) events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following additivity rule:

$$P_{rob}\left(\bigcup_{j=1}^{N} A_j\right) = \sum_{j=1}^{N} P_{rob}\left(A_j\right)$$

And we say also that for N independent events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following product rule:

$$P_{rob}\left(\bigcap_{j=1}^{N} A_j\right) = \prod_{j=1}^{N} P_{rob}\left(A_j\right)$$

III-2- Adding the Imaginary Part $\mathcal{M}$

Now, we can add to this system of axioms an imaginary part such that:

Axiom 6: Let $P_m = i \times (1 - P_r)$ be the probability of an associated event in $\mathcal{M}$ (the imaginary part) to the event A in $\mathcal{R}$ (the real part). It follows that $P_r + P_m / i = 1$ where i is the imaginary number with $i = \sqrt{-1}$ or $i^2 = -1$.

Axiom 7: We construct the complex number or vector $Z = P_r + P_m = P_r + i(1 - P_r)$ having a norm $|Z|$ such that:

$$|Z|^2 = P_r^2 + (P_m / i)^2 .$$

Axiom 8: Let *Pc* denote the probability of an event in the complex probability universe $\mathcal{C}$ where $\mathcal{C} = \mathcal{R} + \mathcal{M}$. We say that *Pc* is the probability of an event *A* in $\mathcal{R}$ with its associated event in $\mathcal{M}$ such that:

$$Pc^2 = (P_r + P_m / i)^2 = |Z|^2 - 2iP_rP_m \quad \text{and is always equal to 1.}$$

We can see that by taking into consideration the set of imaginary probabilities we added three new and original axioms and consequently the system of axioms defined by Kolmogorov was hence expanded to encompass the set of imaginary numbers.

III-3- The Purpose of Extending the Axioms

After adding the new three axioms, it becomes clear that the addition of the imaginary dimensions to the real stochastic experiment yields a probability always equal to one in the complex probability set $\mathcal{C}$. Actually, we will understand directly this result when we realize that the set of probabilities is formed now of two parts: the first part is real and the second part is imaginary. The stochastic event that is happening in the set $\mathcal{R}$ of real probabilities (like in the experiment of coin tossing and getting a tail or a head) has a corresponding real probability P_r and a corresponding imaginary probability P_m. In addition, let $\mathcal{M}$ be the set of imaginary probabilities and let $|Z|^2$ be the Degree of Our Knowledge (*DOK* for short) of this experiment. According to the axioms of Kolmogorov, P_r is always the probability of the phenomenon in the set $\mathcal{R}$. [97-102]

- In fact, a total ignorance of the set $\mathcal{M}$ leads to:
  $P_{rob}(\text{event}) = P_r = 0.5$, P_m = P_{rob}(imaginary part) = 0.5*i*, and $|Z|^2 = DOK$ in this case is equal to: $1 - 2P_r(1 - P_r) = 1 - (2 \times 0.5) \times (1 - 0.5) = 0.5 = 50\%$

- Conversely, a total knowledge of the set in $\mathcal{R}$ leads to:
  $P_{rob}(\text{event}) = P_r = 1$ and P_m = P_{rob}(imaginary part) = 0. Here we have $DOK = 1 - (2 \times 1) \times (1 - 1) = 1$ because the phenomenon is totally known, that is, all the variables and laws affecting the experiment are determined completely, therefore; our degree of our knowledge (*DOK*) of the system is 1 = 100%.

- Now, if we are for sure that an event will never happen i.e. like 'getting nothing' (the empty set), P_r is accordingly = 0, that is the event will never occur in $\mathcal{R}$. P_m will be equal to:
  $i(1 - P_r) = i(1 - 0) = i$, and $|Z|^2 = DOK = 1 - (2 \times 0) \times (1 - 0) = 1$, because we are sure that the event of getting nothing will never happen; therefore, the Degree of Our Knowledge (*DOK*) of the system is 1 = 100%.

We can deduce that we have always:

$$0.5 \le |Z|^2 \le 1, \quad \forall P_r: \ 0 \le P_r \le 1$$

$$\text{and } |Z|^2 = DOK = P_r^2 + (P_m / i)^2 \text{, where } 0 \le P_r, P_m / i \le 1$$

And what is crucial is that in all cases we have:

$$Pc^2 = (P_r + P_m / i)^2 = |Z|^2 - 2iP_rP_m = \left[P_r + (1 - P_r)\right]^2 = 1^2 = 1$$

Actually, according to an experimenter in $\mathcal{R}$, the phenomenon is random: the experimenter ignores the outcome of the chaotic phenomenon. Each outcome will be assigned a probability P_r and he will say that the outcome is nondeterministic. But in the complex probability universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$, the outcome of the random phenomenon will be totally predicted by the observer since the contributions of the set $\mathcal{M}$ were taken into consideration, so this will give:

$$Pc^2 = (P_r + P_m / i)^2$$

Therefore Pc is always equal to 1. Actually, adding the imaginary set to our stochastic phenomenon leads to the elimination of randomness, of ignorance, and of nondeterminism. Subsequently, conducting experiments of this class of phenomena in the set $\mathcal{C}$ is of great importance since we will be able to foretell with certainty the output of all random phenomenon. In fact, conducting experiments in the set $\mathcal{R}$ leads to uncertainty and unpredictability. So, we place ourselves in the set $\mathcal{C}$ instead of placing ourselves in the set $\mathcal{R}$, then study the random events, since in $\mathcal{C}$ we take into consideration all the contributions of the set $\mathcal{M}$ and therefore a deterministic study of the stochastic experiment becomes possible. Conversely, by taking into consideration the contributions of the probability set $\mathcal{M}$ we place ourselves in the set $\mathcal{C}$ and by disregarding $\mathcal{M}$ we restrict our experiment to nondeterministic events in $\mathcal{R}$. [103-112]

Furthermore, we can deduce from the above axioms and definitions that:

$$\begin{aligned} 2iP_rP_m &= 2i \times P_r \times i \times (1 - P_r) \\ &= 2i^2 \times P_r \times (1 - P_r) = -2P_r(1 - P_r) \\ &= Chf \end{aligned}$$

$2iP_rP_m$ will be called the Chaotic factor in our stochastic event and will be denoted accordingly by '*Chf*'. We will understand why we have named this term the chaotic factor; in fact:

- In case $P_r = 1$, that means in the case of a certain event, then the chaotic factor of the event is equal to 0.
- In case $P_r = 0$, that means in the case of an impossible event, then *Chf* = 0. Therefore, in both two last cases, there is no chaos because the output of the event is certain and is known in advance.
- In case $P_r = 0.5$, $Chf = -0.5$.

So, we deduce that: $-0.5 \le Chf \le 0, \quad \forall P_r: \ 0 \le P_r \le 1$. (Figures 2-4)

Consequently, what is truly interesting here is therefore we have quantified both the degree of our knowledge and the chaotic factor of any stochastic phenomenon and hence we can state accordingly:

$$Pc^2 = |Z|^2 - 2iP_rP_m = DOK - Chf$$

Then we can conclude that:

Pc^2 = Degree of our knowledge of the system – Chaotic factor = 1,

therefore $Pc = 1$ permanently and constantly.

This directly leads to the following crucial conclusion: if we succeed to subtract and eliminate the chaotic factor in any stochastic phenomenon, then we will have the outcome probability always equal to one. [37-90] [113-122]

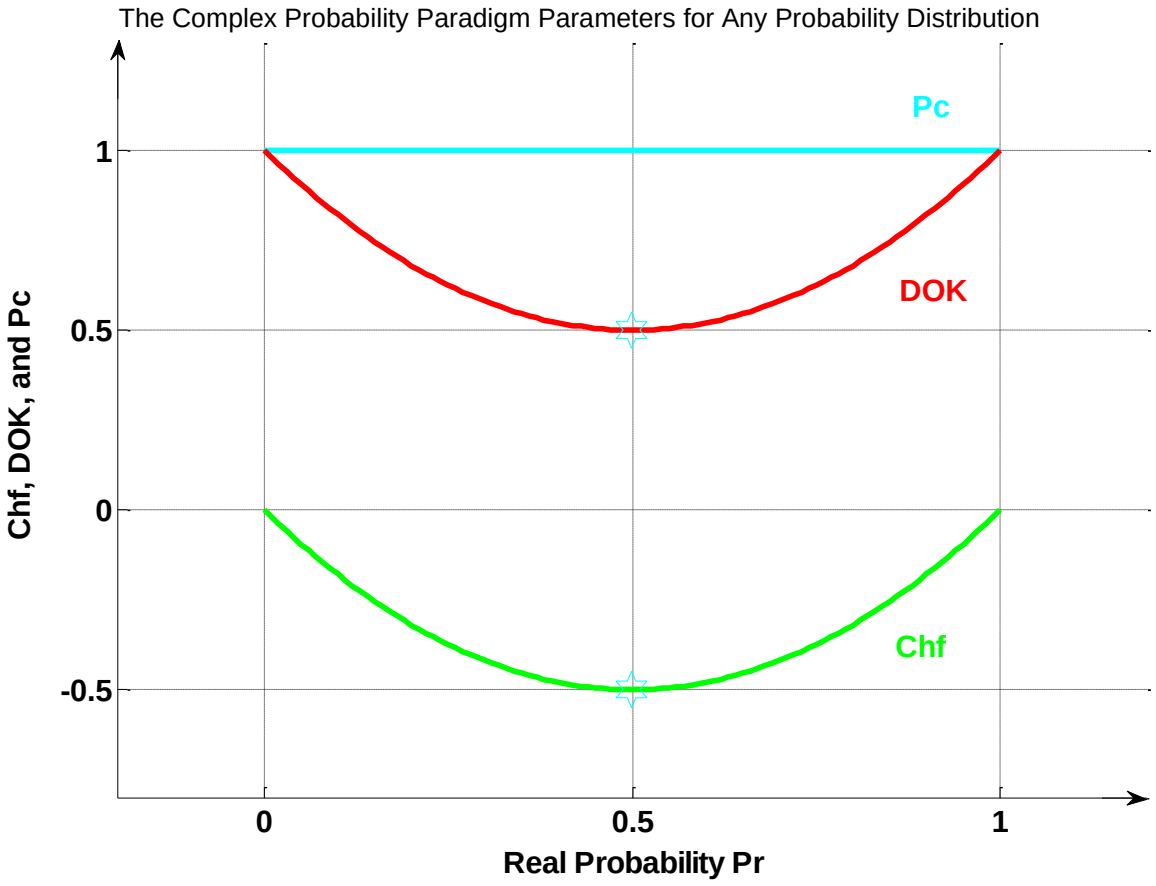


Figure 2. *Chf*, *DOK*, and *Pc* for any probability distribution in 2*D*

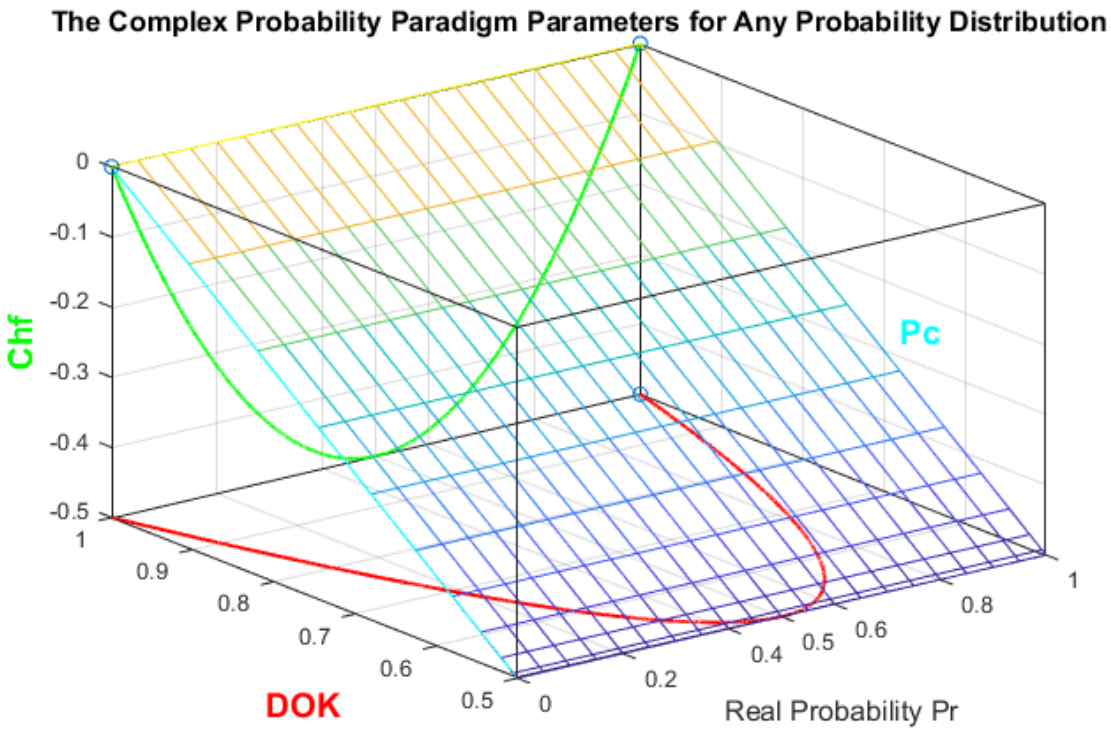


Figure 3. *DOK*, *Chf*, and *Pc* for any probability distribution in 3*D* with $Pc^2 = DOK - Chf = 1 = Pc$

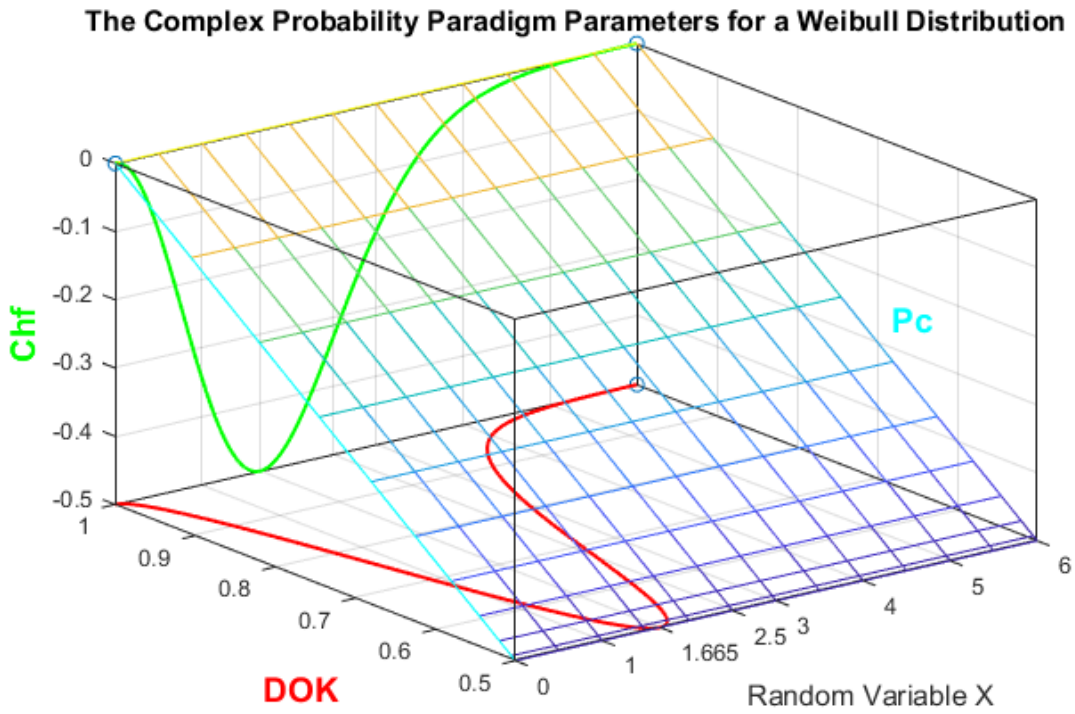


Figure 4. *DOK*, *Chf*, and *Pc* for a Weibull probability distribution in 3*D* with $Pc^2 = DOK - Chf = 1 = Pc$

The graph below illustrates the linear relation between both *DOK* and *Chf*. (Figure 5)

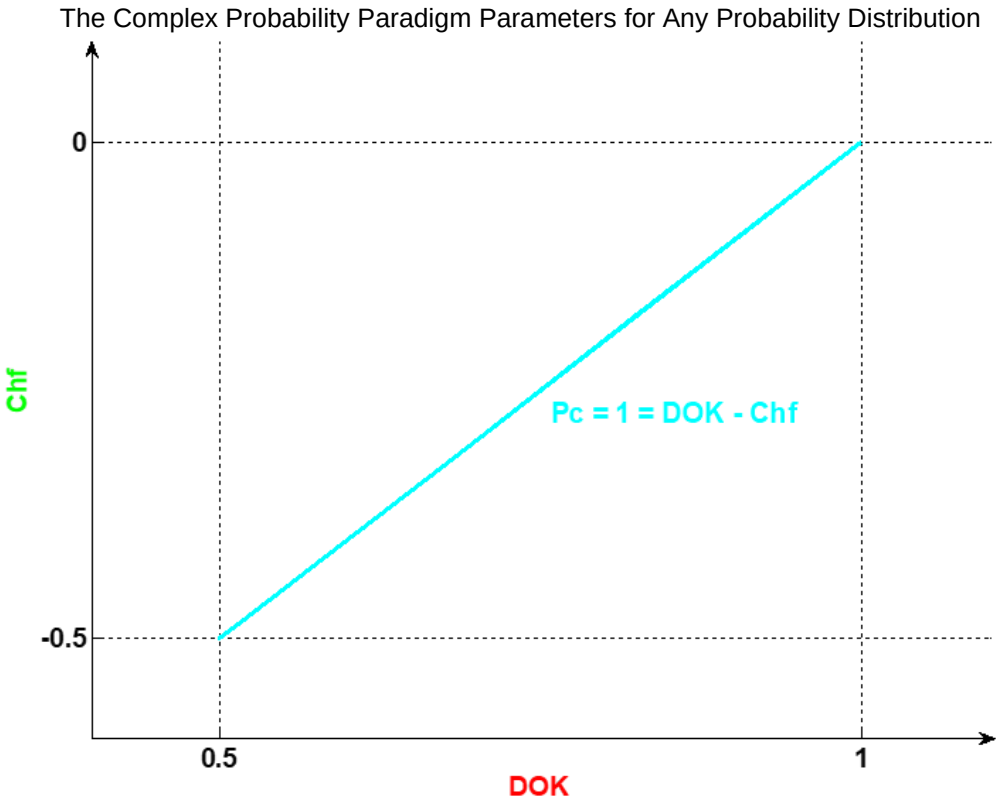


Figure 5. Graph of $Pc^2 = DOK - Chf = 1 = Pc$ for any probability distribution

Furthermore, we require in our present analysis the absolute value of the chaotic factor that will quantify for us the magnitude of the chaotic and stochastic influences on the random system considered which is materialized by the real probability P_r and a probability density function, and which lead to an increasing or decreasing system chaos in $\mathcal{R}$. This additional and original term will be denoted accordingly *MChf* or Magnitude of the Chaotic factor. Therefore, we define this new term by:

$$MChf = |Chf| = |2iP_rP_m| = -2iP_rP_m = 2P_r(1-P_r) \geq 0, \quad \forall P_r: \ 0 \leq P_r \leq 1,$$

And

$$\begin{aligned} Pc^2 &= DOK - Chf \\ &= DOK + |Chf| \ , \quad \text{since} \ \ -0.5 \leq Chf \leq 0 \\ &= DOK + MChf = 1, \end{aligned}$$

$$\Leftrightarrow 0 \leq MChf \leq 0.5 \ \text{ where } \ 0.5 \leq DOK \leq 1.$$

The graph below (Figure 6) illustrates the linear relation between both *DOK* and *MChf*. Moreover, Figures 7 to 13 illustrate the graphs of *Chf*, *MChf*, *DOK*, and *Pc* as functions of the real probability P_r and of the random variable X for any probability distribution and for a Weibull probability distribution. [37-90]

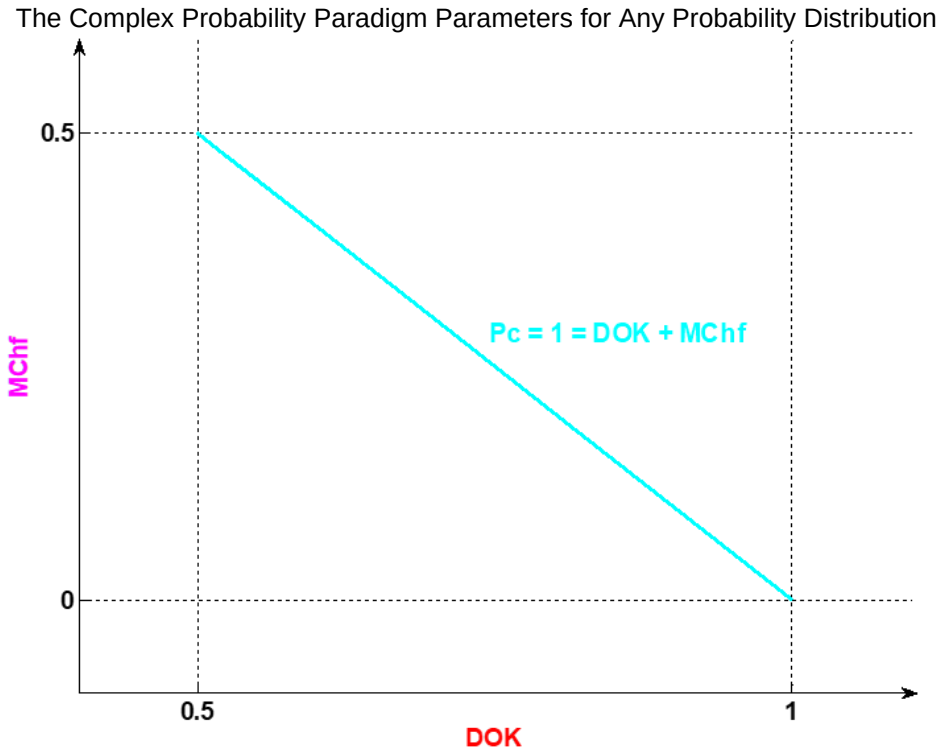


Figure 6. Graph of $Pc^2 = DOK + MChf = 1 = Pc$ for any probability distribution

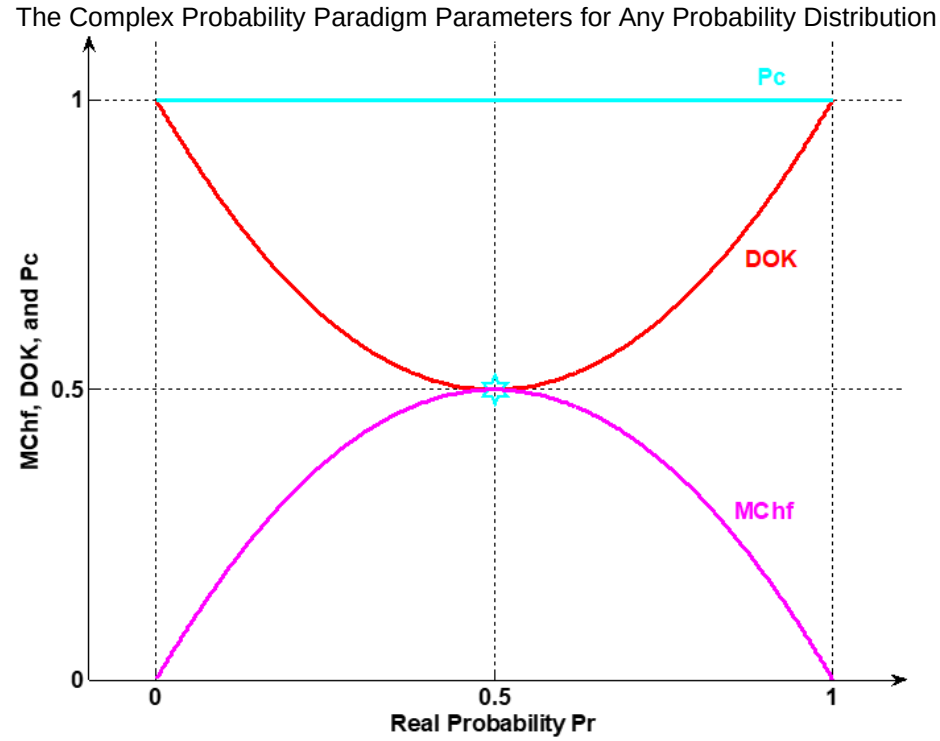


Figure 7. *MChf*, *DOK*, and *Pc* for any probability distribution in 2*D*

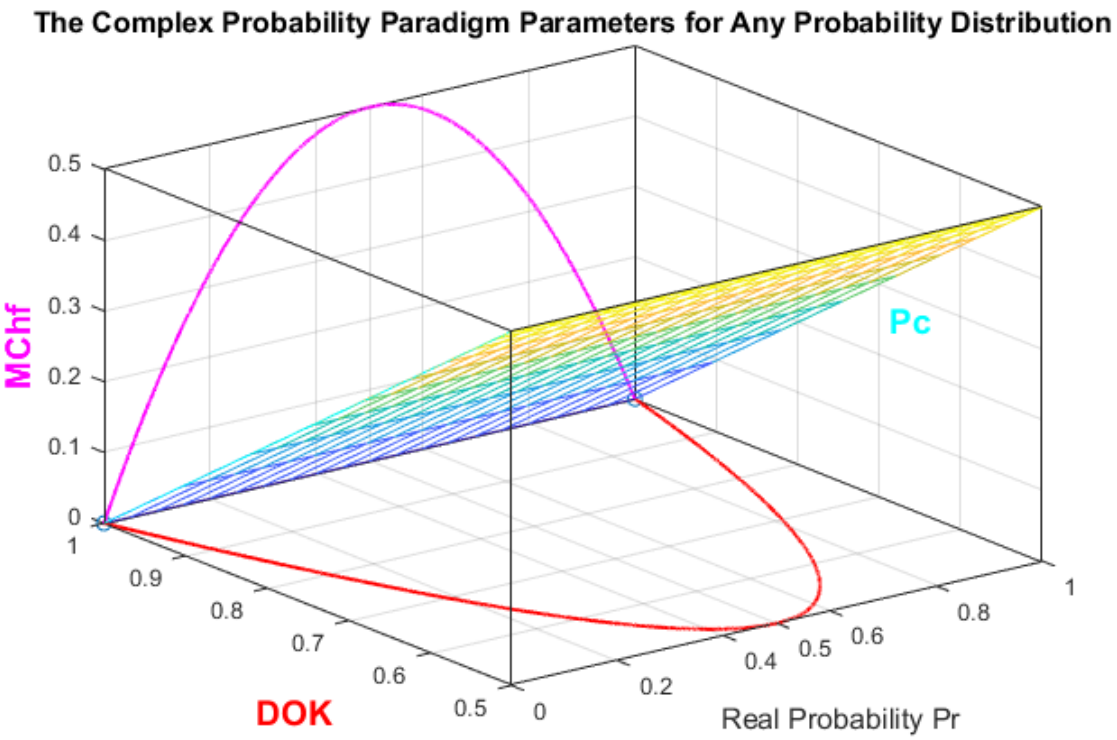


Figure 8. *DOK*, *MChf*, and *Pc* for any probability distribution in 3*D* with

$$Pc^2 = DOK + MChf = 1 = Pc$$

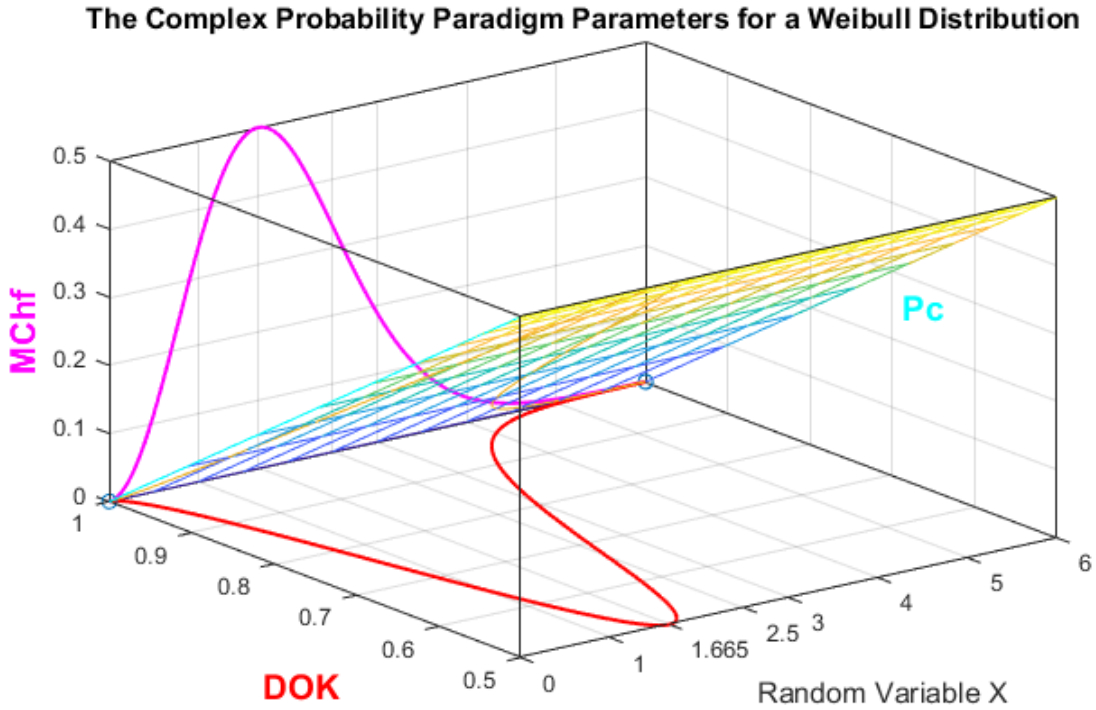


Figure 9. *DOK*, *MChf*, and *Pc* for a Weibull probability distribution in 3*D* with

$$Pc^2 = DOK + MChf = 1 = Pc$$

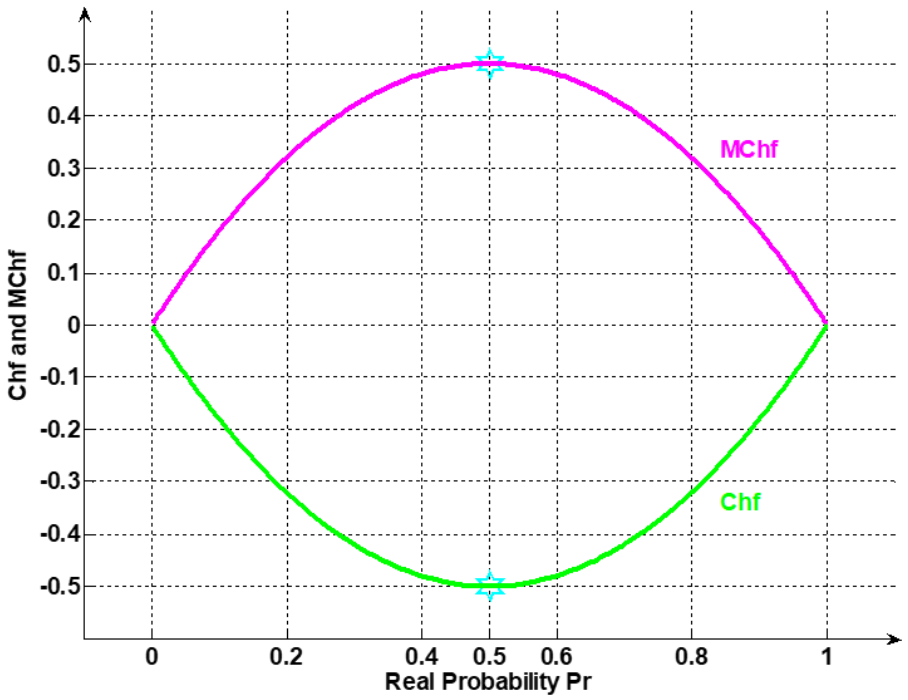


Figure 10. *Chf* and *MChf* for any probability distribution in 2*D*

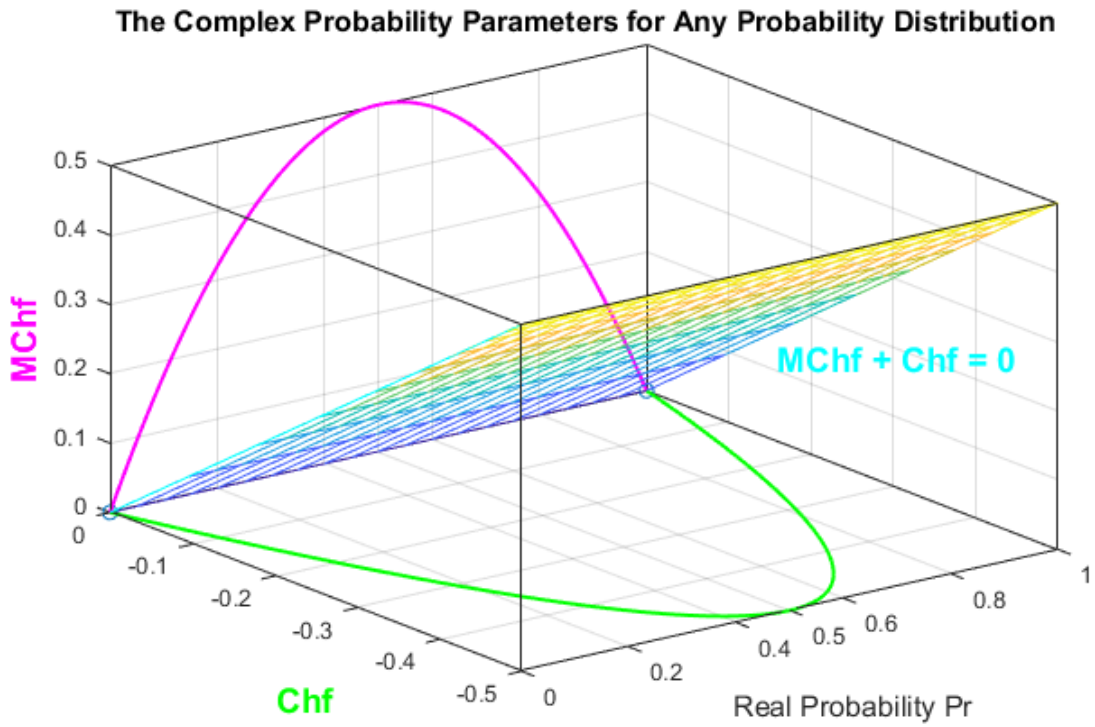


Figure 11. *Chf* and *MChf* for any probability distribution in 3*D* with *MChf* + *Chf* = 0

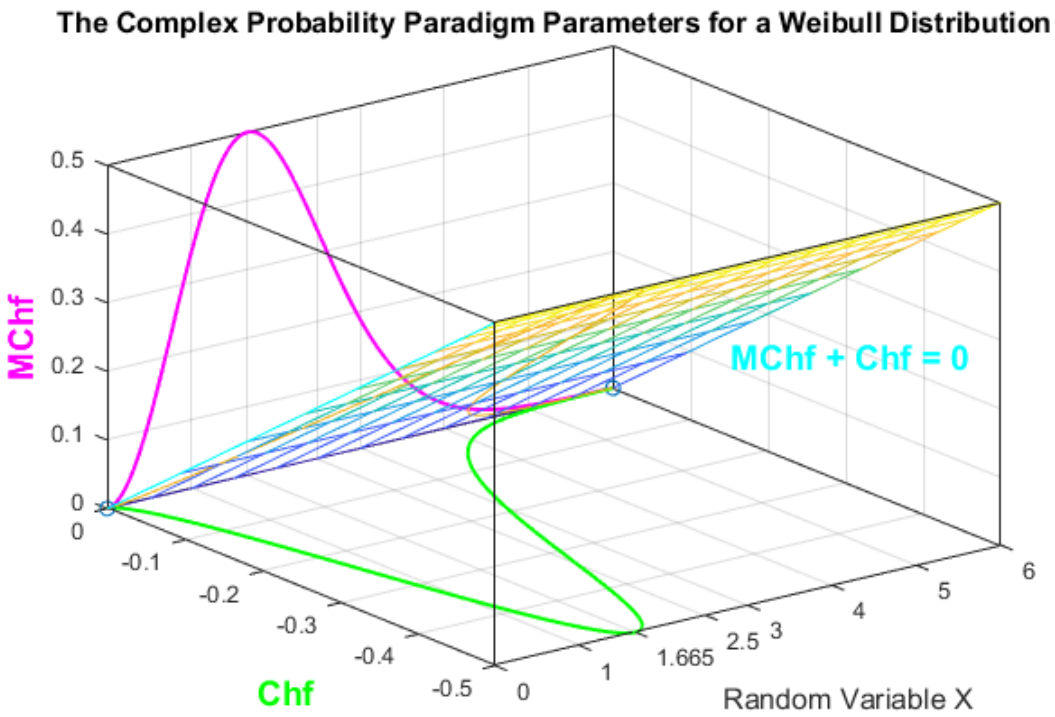


Figure 12. *Chf* and *MChf* for a Weibull probability distribution in 3*D* with *MChf* + *Chf* = 0

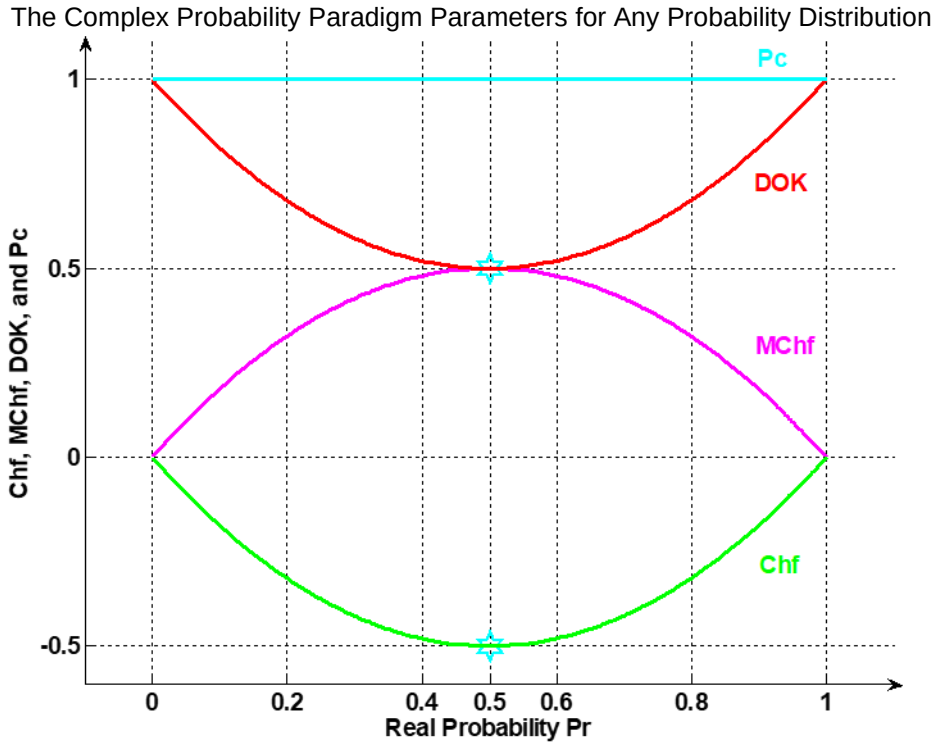


Figure 13. *Chf*, *MChf*, *DOK*, and *Pc* for any probability distribution in 2*D*

To conclude and to summarize, in the real probability universe $\mathcal{R}$ our degree of our certain knowledge is regrettably imperfect, therefore we extend our study to the complex set $\mathcal{C}$ which embraces the contributions of both the real probabilities set $\mathcal{R}$ and the imaginary probabilities set $\mathcal{M}$. Subsequently, this will lead to a perfect and complete degree of knowledge in the universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ (since $Pc = 1$). In fact, working in the complex universe $\mathcal{C}$ leads to a certain prediction of any random event, because in $\mathcal{C}$ we eliminate and subtract from the calculated degree of our knowledge the quantified chaotic factor. This will yield a probability in the universe $\mathcal{C}$ equal to one ($Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$). Many illustrations considering various continuous and discrete probability distributions in my previous research papers verify this hypothesis and novel paradigm. [37-90] The Extended Kolmogorov Axioms (*EKA* for short) or the Complex Probability Paradigm (*CPP* for short) can be summarized and shown in the following figure (Figure 14):

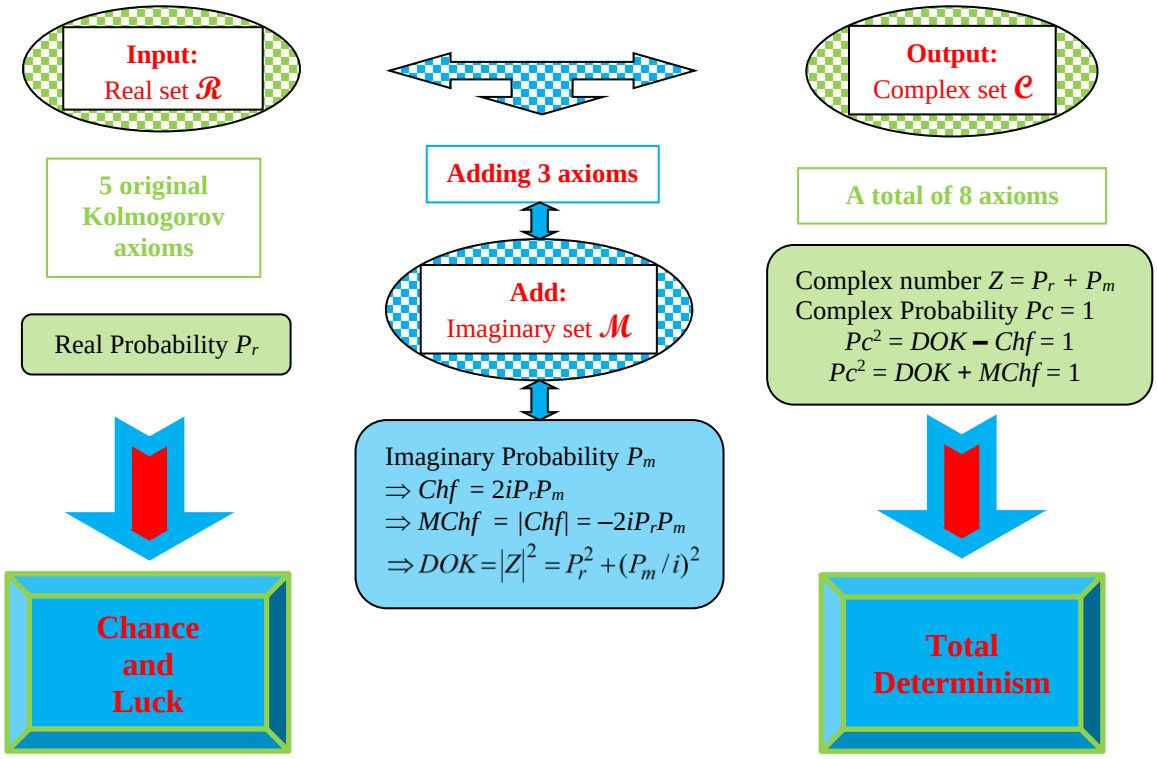


Figure 14- The *EKA* or the *CPP* diagram

IV- The Monte Carlo Techniques of Integration and Simulation [123-133]

In applied mathematics, the name *Monte Carlo* is given to the method of solving problems by means of experiments with random numbers. This name, after the casino at Monaco, was first applied around 1944 to the method of solving deterministic problems by reformulating them in terms of a problem with random elements which could then be solved by large-scale sampling. But, by extension, the term has come to mean any simulation that uses random numbers.

The development and proliferation of computers has led to the widespread use of Monte Carlo methods in virtually all branches of science, ranging from nuclear physics (where computer-aided Monte Carlo was first applied) to astrophysics, biology, engineering, medicine, operations research, and the social sciences.

The Monte Carlo Method of solving problems by using random numbers in a computer – either by direct simulation of physical or statistical problems or by reformulating deterministic problems in terms of one incorporating randomness – has become one of the most important tools of applied mathematics and computer science. A significant proportion of articles in technical journals in such fields as physics, chemistry, and statistics contain articles reporting results of Monte Carlo simulations or suggestions on how they might be applied. Some journals are devoted almost entirely to Monte Carlo problems in their fields. Studies in the formation of the universe or of stars and their planetary systems use Monte Carlo techniques. Studies in genetics, the biochemistry of DNA, and the random configuration and knotting of biological molecules are studied by Monte Carlo methods. In number theory, Monte Carlo methods play an important role in determining primality or factoring of very large integers far beyond the range of deterministic methods. Several important new statistical techniques such as “bootstrapping” and “jackknifing” are based on Monte Carlo methods.

Hence, the role of Monte Carlo methods and simulation in all of the sciences has increased in importance during the past several years. These methods play a central role in the rapidly developing subdisciplines of the computational physical sciences, the computational life sciences, and the other computational sciences. Therefore, the growing power of computers and the evolving simulation methodology have led to the recognition of computation as a third approach for advancing the natural sciences, together with theory and traditional experimentation. At the kernel of Monte Carlo simulation is random number generation.

Now we turn to the approximation of a definite integral by the Monte Carlo method. If we select the first N elements $x_1, x_2, \ldots, x_N$ from a random sequence in the interval (0,1), then:

$$\int_0^1 f(x).dx \cong \frac{(1-0)}{N}\sum_{j=1}^{N} f(x_j) = \frac{1}{N}\sum_{j=1}^{N} f(x_j)$$

Here the integral is approximated by the average of N numbers $f(x_1), f(x_2), \ldots, f(x_N)$. When this is actually carried out, the error is of order $\frac{1}{\sqrt{N}}$, which is not at all competitive with good algorithms, such as the *Romberg* method. However, in higher dimensions, the Monte Carlo method can be quite attractive. For example,

$$\int_0^1\int_0^1\int_0^1 f(x,y,z).dx.dy.dz \cong \frac{[(1-0)\times(1-0)\times(1-0)]}{N}\sum_{j=1}^{N} f(x_j,y_j,z_j) = \frac{1}{N}\sum_{j=1}^{N} f(x_j,y_j,z_j)$$

where (x_j, y_j, z_j) is a random sequence of *N* points in the unit cube $0 \le x \le 1,\ 0 \le y \le 1$, and $0 \le z \le 1$. To obtain random points in the cube, we assume that we have a random sequence in (0,1) denoted by $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \ldots$ To get our first random point p_1 in the cube, just let $p_1 = (\xi_1, \xi_2, \xi_3)$. The second is, of course, $p_2 = (\xi_4, \xi_5, \xi_6)$ and so on.

If the interval (in a one-dimensional integral) is not of length 1, but say is the general case (*a*, *b*), then the average of *f* over *N* random points in (*a*, *b*) is not simply an approximation for the integral but rather for:

$$\frac{1}{b-a}\int_a^b f(x).dx$$

which agrees with our intention that the function $f(x) = 1$ has an average of 1. Similarly, in higher dimensions, the average of *f* over a region is obtained by integrating and dividing by the area, volume, or measure of that region. For instance,

$$\frac{1}{[(7-4)\times(5-(-2))\times(3-0)]}\int_4^7\int_{-2}^5\int_0^3 f(x,y,z).dx.dy.dz = \frac{1}{63}\int_4^7\int_{-2}^5\int_0^3 f(x,y,z).dx.dy.dz$$

is the average of *f* over the parallelepiped described by the following three inequalities:

$$0 \le x \le 3, -2 \le y \le 5,\ 4 \le z \le 7.$$

To keep the limits of integration straight, we recall that:

$$\int_a^b\int_c^d f(x,y).dx.dy = \int_a^b\left[\int_c^d f(x,y).dx\right].dy$$

and

$$\int_{a1}^{a2}\int_{b1}^{b2}\int_{c1}^{c2} f(x,y,z).dx.dy.dz = \int_{a1}^{a2}\left\{\int_{b1}^{b2}\left[\int_{c1}^{c2} f(x,y,z).dx\right].dy\right\}.dz$$

So, if (x_j, y_j) denote random points with appropriate uniform distribution, the following examples illustrate Monte Carlo techniques:

$$\int_1^9 f(x).dx \cong \frac{(9-1)}{N}\sum_{j=1}^N f(x_j) = \frac{8}{N}\sum_{j=1}^N f(x_j)$$

$$\int_4^8\int_2^5 f(x,y).dx.dy \cong \frac{[(8-4)\times(5-2)]}{N}\sum_{j=1}^N f(x_j,y_j) = \frac{12}{N}\sum_{j=1}^N f(x_j,y_j)$$

In each case, the random points should be uniformly distributed in the regions involved.

In general, we have:

$$\int_A f \cong (\text{measure of } A) \times (\text{average of } f \text{ over } N \text{ random points in } A)$$

Here we are using the fact that the average of a function on a set is equal to the integral of the function over the set divided by the measure of the set.

V- The Complex Probability Paradigm and Monte Carlo Methods Parameters [37-90] [134-141]

V-1- The Probabilities of Convergence and Divergence

Let R_E be the exact result of the random experiment or of a simple or a multidimensional integral that are not always possible to evaluate by ordinary methods of probability theory or calculus or deterministic numerical methods. And let R_A be the approximate result of these experiments and integrals found by Monte Carlo methods.

The relative error in the Monte Carlo methods is: Rel. Error $= \left|\frac{R_E - R_A}{R_E}\right| = \left|1 - \frac{R_A}{R_E}\right|$

In addition, the percent relative error is = $100\% \times \left|\frac{R_E - R_A}{R_E}\right|$ and is always between 0% and 100%.

Therefore, the relative error is always between 0 and 1. Hence:

$$0 \le \left|\frac{R_E - R_A}{R_E}\right| \le 1 \Leftrightarrow \begin{cases} 0 \le \left(\frac{R_E - R_A}{R_E}\right) \le 1 & \text{if } R_A \le R_E \\ 0 \le -\left(\frac{R_E - R_A}{R_E}\right) \le 1 & \text{if } R_A \ge R_E \end{cases} \Leftrightarrow \begin{cases} 0 \le R_A \le R_E \\ R_E \le R_A \le 2R_E \end{cases}$$

Moreover, we define the real probability by:

$$P_r = 1 - \left|\frac{R_E - R_A}{R_E}\right| = 1 - \left|1 - \frac{R_A}{R_E}\right| = \begin{cases} 1 - \left(1 - \frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ 1 + \left(1 - \frac{R_A}{R_E}\right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = \begin{cases} \frac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ 2 - \frac{R_A}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

= 1 – the relative error in the Monte Carlo method
= Probability of Monte Carlo method convergence in $\mathcal{R}$.

And therefore:

$$P_m = i\left(1-P_r\right) = i\left\{1-\left[1-\left|\frac{R_E-R_A}{R_E}\right|\right]\right\} = i\left\{1-\left[1-\left|1-\frac{R_A}{R_E}\right|\right]\right\} = i\left|1-\frac{R_A}{R_E}\right|$$

$$= \begin{cases} i\left(1-\dfrac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ -i\left(1-\dfrac{R_A}{R_E}\right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = \begin{cases} i\left(1-\dfrac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ i\left(\dfrac{R_A}{R_E}-1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

= Probability of Monte Carlo method divergence in the imaginary probability set $\mathcal{M}$ since it is the imaginary complement of P_r.

Consequently,

$$P_m / i = 1-P_r = \left|1-\frac{R_A}{R_E}\right| = \begin{cases} 1-\dfrac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ \dfrac{R_A}{R_E}-1 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

= The relative error in the Monte Carlo method
= Probability of Monte Carlo method divergence in $\mathcal{R}$ since it is the real complement of P_r.

In the case where $0 \le R_A \le R_E$ we have $0 \le \frac{R_A}{R_E} \le 1 \Rightarrow 0 \le P_r \le 1$ and we deduce also that $0 \le \left(1-\frac{R_A}{R_E}\right) \le 1 \Rightarrow 0 \le P_m / i \le 1$ and $\Rightarrow 0 \le P_m \le i$

And in the case where $R_E \le R_A \le 2R_E \Rightarrow 1 \le \frac{R_A}{R_E} \le 2 \Rightarrow 0 \le \left(2-\frac{R_A}{R_E}\right) \le 1 \Rightarrow 0 \le P_r \le 1$ and we deduce also that $0 \le \left(\frac{R_A}{R_E}-1\right) \le 1 \Rightarrow 0 \le P_m / i \le 1$ and $\Rightarrow 0 \le P_m \le i$

Therefore, if $R_A = 0$ or $R_A = 2R_E$ that means before the beginning of the simulation, then:

$P_r = P_{rob}$ (convergence) in $\mathcal{R} = 0$
$P_m = P_{rob}$ (divergence) in $\mathcal{M} = i$
$P_m / i = P_{rob}$ (divergence) in $\mathcal{R} = 1$

And if $R_A = R_E$ that means at the end of Monte Carlo simulation then:

$P_r = P_{rob}$ (convergence) in $\mathcal{R} = 1$
$P_m = P_{rob}$ (divergence) in $\mathcal{M} = 0$
$P_m / i = P_{rob}$ (divergence) in $\mathcal{R} = 0$

V-2- The Complex Random Vector Z in $\mathcal{C}$

We have $Z = P_r + P_m = \begin{cases} \frac{R_A}{R_E} + i\left(1 - \frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ \left(2 - \frac{R_A}{R_E}\right) + i\left(\frac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$

where $\mathrm{Re}(Z) = P_r = \begin{cases} \frac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ 2 - \frac{R_A}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases}$ = the real part of Z

and $\mathrm{Im}(Z) = P_m / i = \begin{cases} 1 - \frac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ \frac{R_A}{R_E} - 1 & \text{if } R_E \le R_A \le 2R_E \end{cases}$ = the imaginary part of Z.

That means that the complex random vector Z is the sum in $\mathcal{C}$ of the real probability of convergence in $\mathcal{R}$ and of the imaginary probability of divergence in $\mathcal{M}$.

If $R_A = 0$ (before the simulation begins) then $P_r = \frac{R_A}{R_E} = 0$ and $P_m = i\left(1 - \frac{R_A}{R_E}\right) = i(1-0) = i$

therefore $Z = 0 + i = i$.

If $R_A = \frac{R_E}{2}$ or $R_A = \frac{3R_E}{2}$ (at the middle of the simulation) then:

$$P_r = \begin{cases} \frac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ 2 - \frac{R_A}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases} = \begin{cases} \frac{R_E}{2R_E} = 0.5 & \text{if } 0 \le R_A \le R_E \\ 2 - \frac{3R_E}{2R_E} = 0.5 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow P_r = 0.5$$

and $P_m = \begin{cases} i\left(1 - \frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ i\left(\frac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = \begin{cases} i\left(1 - \frac{R_E}{2R_E}\right) = 0.5i & \text{if } 0 \le R_A \le R_E \\ i\left(\frac{3R_E}{2R_E} - 1\right) = 0.5i & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow P_m = 0.5i$

therefore $Z = 0.5 + 0.5i$.

If $R_A = R_E$ (at the simulation end) then:

$$P_r = \begin{cases} \frac{R_A}{R_E} = \frac{R_E}{R_E} = 1 & \text{if } 0 \le R_A \le R_E \\ 2 - \frac{R_A}{R_E} = 2 - \frac{R_E}{R_E} = 2 - 1 = 1 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow P_r = 1$$

And

$$P_m=\begin{cases} i\left(1-\dfrac{R_A}{R_E}\right) & \text{if } 0\le R_A\le R_E \\ i\left(\dfrac{R_A}{R_E}-1\right) & \text{if } R_E\le R_A\le 2R_E \end{cases}=\begin{cases} i\left(1-\dfrac{R_E}{R_E}\right) & \text{if } 0\le R_A\le R_E \\ i\left(\dfrac{R_E}{R_E}-1\right) & \text{if } R_E\le R_A\le 2R_E \end{cases}=\begin{cases} 0 & \text{if } 0\le R_A\le R_E \\ 0 & \text{if } R_E\le R_A\le 2R_E \end{cases}$$

$$\Leftrightarrow P_m=0$$

therefore $Z=1+0i=1$.

V-3- The Degree of Our Knowledge *DOK*

We have:

$$DOK=|Z|^2=P_r^2+(P_m/i)^2=\begin{cases} \left(\dfrac{R_A}{R_E}\right)^2 & \text{if } 0\le R_A\le R_E \\ \left(2-\dfrac{R_A}{R_E}\right)^2 & \text{if } R_E\le R_A\le 2R_E \end{cases}+\begin{cases} \left(1-\dfrac{R_A}{R_E}\right)^2 & \text{if } 0\le R_A\le R_E \\ \left(\dfrac{R_A}{R_E}-1\right)^2 & \text{if } R_E\le R_A\le 2R_E \end{cases}$$

$$=\begin{cases} \left(\dfrac{R_A}{R_E}\right)^2+\left(1-\dfrac{R_A}{R_E}\right)^2 & \text{if } 0\le R_A\le R_E \\ \left(2-\dfrac{R_A}{R_E}\right)^2+\left(\dfrac{R_A}{R_E}-1\right)^2 & \text{if } R_E\le R_A\le 2R_E \end{cases}=\begin{cases} 2\left(\dfrac{R_A}{R_E}\right)^2-2\left(\dfrac{R_A}{R_E}\right)+1 & \text{if } 0\le R_A\le R_E \\ 2\left(\dfrac{R_A}{R_E}\right)^2-6\left(\dfrac{R_A}{R_E}\right)+5 & \text{if } R_E\le R_A\le 2R_E \end{cases}$$

From *CPP* we have that $0.5\le DOK\le 1$ then if $DOK=0.5$

$$\Leftrightarrow\begin{cases} 2\left(\dfrac{R_A}{R_E}\right)^2-2\left(\dfrac{R_A}{R_E}\right)+1=0.5 & \text{if } 0\le R_A\le R_E \\ 2\left(\dfrac{R_A}{R_E}\right)^2-6\left(\dfrac{R_A}{R_E}\right)+5=0.5 & \text{if } R_E\le R_A\le 2R_E \end{cases}$$

then solving the second-degree equation for $\dfrac{R_A}{R_E}$ gives:

$$\begin{cases} \dfrac{R_A}{R_E}=1/2 & \text{if } 0\le R_A\le R_E \\ \dfrac{R_A}{R_E}=3/2 & \text{if } R_E\le R_A\le 2R_E \end{cases}\Leftrightarrow\begin{cases} R_A=R_E/2 & \text{if } 0\le R_A\le R_E \\ R_A=3R_E/2 & \text{if } R_E\le R_A\le 2R_E \end{cases}$$ and vice versa.

That means that *DOK* is minimum when the approximate result is equal to half of the exact result if $0 \le R_A \le R_E$ or when the approximate result is equal to three times the half of the exact result if $R_E \le R_A \le 2R_E$, that means at the middle of the simulation.

In addition, if $DOK = 1$ then:

$$\Leftrightarrow \begin{cases} 2\left(\frac{R_A}{R_E}\right)^2 - 2\left(\frac{R_A}{R_E}\right) + 1 = 1 & \text{if } 0 \le R_A \le R_E \\ 2\left(\frac{R_A}{R_E}\right)^2 - 6\left(\frac{R_A}{R_E}\right) + 5 = 1 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} \left(\frac{R_A}{R_E}\right)^2 - \left(\frac{R_A}{R_E}\right) = 0 & \text{if } 0 \le R_A \le R_E \\ 2\left(\frac{R_A}{R_E}\right)^2 - 6\left(\frac{R_A}{R_E}\right) + 4 = 0 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$\Leftrightarrow \begin{cases} R_A = 0 \text{ OR } R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = 2R_E \text{ OR } R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases} \text{ and vice versa.}$$

That means that *DOK* is maximum when the approximate result is equal to 0 or $2R_E$ (before the beginning of the simulation) and when it is equal to the exact result (at the end of the simulation). We can deduce that we have perfect and total knowledge of the stochastic experiment before the beginning of Monte Carlo simulation since no randomness was introduced yet, as well as at the end of the simulation after the convergence of the method to the exact result.

V-4- The Chaotic Factor *Chf*

We have:

$$Chf = 2iP_rP_m = 2i \times \begin{cases} \frac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ 2 - \frac{R_A}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases} \times \begin{cases} i\left(1 - \frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ i\left(\frac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

since $i^2 = -1$ then:

$$Chf = \begin{cases} -2\left(\frac{R_A}{R_E}\right)\left(1 - \frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ -2\left(2 - \frac{R_A}{R_E}\right)\left(\frac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

From *CPP* we have that $-0.5 \le Chf \le 0$ then if $Chf = -0.5$

$$\Leftrightarrow \begin{cases} -2\left(\frac{R_A}{R_E}\right)\left(1-\frac{R_A}{R_E}\right) = -0.5 & \text{if } 0 \le R_A \le R_E \\ -2\left(2-\frac{R_A}{R_E}\right)\left(\frac{R_A}{R_E}-1\right) = -0.5 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} R_A = R_E/2 & \text{if } 0 \le R_A \le R_E \\ R_A = 3R_E/2 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

and vice versa.

That means that *Chf* is minimum when the approximate result is equal to half of the exact result if $0 \le R_A \le R_E$ or when the approximate result is equal to three times the half of the exact result if $R_E \le R_A \le 2R_E$, that means at the middle of the simulation.

In addition, if $Chf = 0$ then:

$$\Leftrightarrow \begin{cases} -2\left(\frac{R_A}{R_E}\right)\left(1-\frac{R_A}{R_E}\right) = 0 & \text{if } 0 \le R_A \le R_E \\ -2\left(2-\frac{R_A}{R_E}\right)\left(\frac{R_A}{R_E}-1\right) = 0 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} R_A = 0 \text{ OR } R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = 2R_E \text{ OR } R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

And, conversely, if $\begin{cases} R_A = 0 \text{ OR } R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = 2R_E \text{ OR } R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases}$ then $Chf = 0$.

That means that *Chf* is equal to 0 when the approximate result is equal to 0 or $2R_E$ (before the beginning of the simulation) and when it is equal to the exact result (at the end of the simulation).

V-5- The Magnitude of the Chaotic Factor *MChf*

We have:

$$MChf = |Chf| = -2iP_rP_m = -2i \times \begin{cases} \frac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ 2-\frac{R_A}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases} \times \begin{cases} i\left(1-\frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ i\left(\frac{R_A}{R_E}-1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

since $i^2 = -1$ then:

$$MChf = \begin{cases} 2\left(\frac{R_A}{R_E}\right)\left(1-\frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ 2\left(2-\frac{R_A}{R_E}\right)\left(\frac{R_A}{R_E}-1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

From *CPP* we have that $0 \le MChf \le 0.5$ then if $MChf = 0.5$

$$\Leftrightarrow \begin{cases} 2\left(\frac{R_A}{R_E}\right)\left(1-\frac{R_A}{R_E}\right)=0.5 & \text{if } 0 \le R_A \le R_E \\ 2\left(2-\frac{R_A}{R_E}\right)\left(\frac{R_A}{R_E}-1\right)=0.5 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} R_A = R_E/2 & \text{if } 0 \le R_A \le R_E \\ R_A = 3R_E/2 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

and vice versa.

That means that *MChf* is maximum when the approximate result is equal to half of the exact result if $0 \le R_A \le R_E$ or when the approximate result is equal to three times the half of the exact result if $R_E \le R_A \le 2R_E$, that means at the middle of the simulation. This implies that the magnitude of the chaos (*MChf*) introduced by the random variables used in Monte Carlo method is maximum at the halfway of the simulation.

In addition, if $MChf = 0$ then:

$$\Leftrightarrow \begin{cases} 2\left(\frac{R_A}{R_E}\right)\left(1-\frac{R_A}{R_E}\right)=0 & \text{if } 0 \le R_A \le R_E \\ 2\left(2-\frac{R_A}{R_E}\right)\left(\frac{R_A}{R_E}-1\right)=0 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} R_A = 0 \text{ OR } R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = 2R_E \text{ OR } R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

And, conversely, if $\Leftrightarrow \begin{cases} R_A = 0 \text{ OR } R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = 2R_E \text{ OR } R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases}$ then $MChf = 0$.

That means that *MChf* is minimum and is equal to 0 when the approximate result is equal to 0 or $2R_E$ (before the beginning of the simulation) and when it is equal to the exact result (at the end of the simulation). We can deduce that the magnitude of the chaos in the stochastic experiment is null before the beginning of Monte Carlo simulation since no randomness was introduced yet, as well as at the end of the simulation after the convergence of the method to the exact result when randomness has finished its task in the stochastic Monte Carlo method and experiment.

V-6- The Probability *Pc* in the Probability Set $\mathcal{C} = \mathcal{R} + \mathcal{M}$

We have:

$$Pc^2 = DOK - Chf = DOK + MChf$$

$$= \begin{cases} 2\left(\frac{R_A}{R_E}\right)^2 - 2\left(\frac{R_A}{R_E}\right)+1 & \text{if } 0 \le R_A \le R_E \\ 2\left(\frac{R_A}{R_E}\right)^2 - 6\left(\frac{R_A}{R_E}\right)+5 & \text{if } R_E \le R_A \le 2R_E \end{cases} - \begin{cases} -2\left(\frac{R_A}{R_E}\right)\left(1-\frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ -2\left(2-\frac{R_A}{R_E}\right)\left(\frac{R_A}{R_E}-1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$= \begin{cases} 1 & \text{if } 0 \le R_A \le R_E \\ 1 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow Pc^2 = 1 \quad \text{for } 0 \le R_A \le 2R_E$$

$\Leftrightarrow Pc = 1$ = Probability of convergence in $\mathcal{C}$, therefore:

$$Pc = \begin{cases} \dfrac{R_A}{R_E} = 1 & \text{if } 0 \le R_A \le R_E \\ 2 - \dfrac{R_A}{R_E} = 1 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$\Leftrightarrow R_A = R_E$ for $0 \le R_A \le 2R_E$ continuously in the probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$. This is due to the fact in $\mathcal{C}$ we have subtracted in the equation above the chaotic factor *Chf* from our knowledge *DOK* and therefore we have eliminated chaos caused and introduced by all the random variables and the stochastic fluctuations that lead to approximate results in the Monte Carlo simulation in $\mathcal{R}$. Therefore, since in $\mathcal{C}$ we have always $R_A = R_E$ then the Monte Carlo simulation which is a stochastic method by nature in $\mathcal{R}$ becomes after applying the *CPP* a deterministic method in $\mathcal{C}$ since the probability of convergence of any random experiment in $\mathcal{C}$ is constantly and permanently equal to 1 for any iterations number *N*.

V-7- The Rates of Change of the Probabilities in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$

$$\text{Since } Z = P_r + P_m = \begin{cases} \dfrac{R_A}{R_E} + i\left(1 - \dfrac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ \left(2 - \dfrac{R_A}{R_E}\right) + i\left(\dfrac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = \text{Re}(Z) + i\,\text{Im}(Z)$$

Then:

$$\frac{dZ}{dR_A} = \frac{dP_r}{dR_A} + \frac{dP_m}{dR_A} = \begin{cases} \dfrac{d}{dR_A}\left[\dfrac{R_A}{R_E} + i\left(1 - \dfrac{R_A}{R_E}\right)\right] & \text{if } 0 \le R_A \le R_E \\ \dfrac{d}{dR_A}\left[\left(2 - \dfrac{R_A}{R_E}\right) + i\left(\dfrac{R_A}{R_E} - 1\right)\right] & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$= \begin{cases} \dfrac{d}{dR_A}\left[\dfrac{R_A}{R_E}\right] + \dfrac{d}{dR_A}\left[i\left(1 - \dfrac{R_A}{R_E}\right)\right] & \text{if } 0 \le R_A \le R_E \\ \dfrac{d}{dR_A}\left[2 - \dfrac{R_A}{R_E}\right] + \dfrac{d}{dR_A}\left[i\left(\dfrac{R_A}{R_E} - 1\right)\right] & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$= \begin{cases} \frac{1}{R_E} - \frac{i}{R_E} = \frac{1}{R_E}(1-i) & \text{if } 0 \le R_A \le R_E \\ -\frac{1}{R_E} + \frac{i}{R_E} = \frac{1}{R_E}(i-1) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

Therefore,

- $$\mathrm{Re}\left[\frac{dZ}{dR_A}\right] = \frac{dP_r}{dR_A} = \begin{cases} +\frac{1}{R_E} & \text{if } 0 \le R_A \le R_E \\ -\frac{1}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$= \begin{cases} \text{constant} > 0 & \text{if } 0 \le R_A \le R_E \text{ and } R_E > 0 \\ \text{constant} < 0 & \text{if } R_E \le R_A \le 2R_E \text{ and } R_E > 0 \end{cases}$$

that means that the slope of the probability of convergence in $\mathcal{R}$ or its rate of change is constant and positive if $0 \le R_A \le R_E$, and constant and negative if $R_E \le R_A \le 2R_E$, and it depends only on R_E ; hence, we have a constant increase in P_r (the convergence probability) as a function of the iterations number N as R_A increases from 0 to R_E and as R_A decreases from $2R_E$ to R_E till P_r reaches the value 1 that means till the random experiment converges to R_E.

$$\mathrm{Im}\left[\frac{dZ}{dR_A}\right] = \frac{1}{i}\frac{dP_m}{dR_A} = \frac{d(P_m/i)}{dR_A} = \begin{cases} -\frac{1}{R_E} & \text{if } 0 \le R_A \le R_E \\ +\frac{1}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$= \begin{cases} \text{constant} < 0 & \text{if } 0 \le R_A \le R_E \text{ and } R_E > 0 \\ \text{constant} > 0 & \text{if } R_E \le R_A \le 2R_E \text{ and } R_E > 0 \end{cases}$$

that means that the slopes of the probabilities of divergence in $\mathcal{R}$ and $\mathcal{M}$ or their rates of change are constant and negative if $0 \le R_A \le R_E$, and constant and positive if $R_E \le R_A \le 2R_E$, and they depend only on R_E; hence, we have a constant decrease in P_m/i and P_m (the divergence probabilities) as functions of the iterations number N as R_A increases from 0 to R_E and as R_A decreases from $2R_E$ to R_E till P_m/i and P_m reach the value 0 that means till the random experiment converges to R_E.

Additionally,

$$\left|\frac{dZ}{dR_A}\right|^2 = \left[\frac{dP_r}{dR_A}\right]^2 + \left[\frac{1}{i}\frac{dP_m}{dR_A}\right]^2 = \left[\frac{dP_r}{dR_A}\right]^2 + \left[\frac{d(P_m/i)}{dR_A}\right]^2$$

$$= \begin{cases} \left(\frac{1}{R_E}\right)^2 + \left(-\frac{1}{R_E}\right)^2 & \text{if } 0 \le R_A \le R_E \\ \left(-\frac{1}{R_E}\right)^2 + \left(\frac{1}{R_E}\right)^2 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$\Leftrightarrow \left|\frac{dZ}{dR_A}\right|^2 = \frac{1}{R_E^2} + \frac{1}{R_E^2} = \frac{2}{R_E^2} \quad \text{for } 0 \le R_A \le 2R_E$$

$\Leftrightarrow \left|\frac{dZ}{dR_A}\right| = \frac{\sqrt{2}}{R_E} = \text{constant} > 0$ if $R_E > 0$; that means that the module of the slope of the complex probability vector *Z* in $\mathbb{C}$ or of its rate of change is constant and positive and it depends only on R_E ; hence, we have a constant increase in $\text{Re}(Z)$ and a constant decrease in $\text{Im}(Z)$ as functions of the iterations number *N* and as *Z* goes from (0, *i*) at *N* = 0 till (1,0) at the simulation end; hence, till $\text{Re}(Z) = P_r$ reaches the value 1 that means till the random experiment converges to R_E.

Furthermore, since $Pc^2 = DOK - Chf = DOK + MChf = 1$ then $Pc = 1 =$ Probability of convergence in $\mathbb{C}$ and consequently : $\frac{d(Pc)}{dR_A} = \frac{d(1)}{dR_A} = 0$, that means that *Pc* is constantly equal to 1 for every value of R_A, of R_E, and of the iterations number *N*, that means for any stochastic experiment and for any simulation of Monte Carlo method. So, we conclude that in $\mathbb{C}$ we have complete and perfect knowledge of the random experiment which has become now a deterministic one since the extension in the complex probability plane $\mathbb{C}$ defined by the *CPP* axioms has changed all stochastic variables to deterministic variables.

VI- The Resultant Complex Random Vector *Z* and the Convergence of Monte Carlo Methods [37-90]

A powerful tool will be described in the current section which was developed in my personal previous research papers and which is founded on the concept of a complex random vector that is a vector combining the real and the imaginary probabilities of a random outcome, defined in the three added axioms of *CPP* by the term $z_j = P_{rj} + P_{mj}$. Accordingly, we will define the vector *Z* as the resultant complex random vector which is the sum of all the complex random vectors z_j in the complex probability plane $\mathbb{C}$. This procedure is illustrated by considering first a general Bernoulli distribution, then we will discuss a discrete probability distribution with *N* equiprobable random vectors as a general case. In fact, if *z* represents one output from the uniform distribution *U*, then Z_U represents the whole system of outputs from the uniform distribution *U* that means the whole random distribution in the complex probability plane $\mathbb{C}$. So, it follows directly

that a Bernoulli distribution can be understood as a simplified system with two random outputs (section VI-1), whereas the general case is a random system with *N* random outputs (section VI-2). Afterward, I will prove the convergence of Monte Carlo methods using this new powerful concept (section VI-3).

VI-1- The Resultant Complex Random Vector *Z* of a General Bernoulli Distribution (A Distribution with Two Random Outputs)

First, let us consider the following general Bernoulli distribution and let us define its complex random vectors and their resultant (Table 1):

Outcome	x_j	x_1	x_2
In $\mathcal{R}$	P_{rj}	$P_{r1}=p$	$P_{r2}=q$
In $\mathcal{M}$	P_{mj}	$P_{m1}=i(1-p)=iq$	$P_{m2}=i(1-q)=ip$
In $\mathcal{C}=\mathcal{R}+\mathcal{M}$	z_j	$z_1=P_{r1}+P_{m1}$	$z_2=P_{r2}+P_{m2}$

Table 1. A general Bernoulli distribution in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$

Where,

x_1 and x_2 are the outcomes of the first and second random vectors respectively.

P_{r1} and P_{r2} are the real probabilities of x_1 and x_2 respectively.

P_{m1} and P_{m2} are the imaginary probabilities of x_1 and x_2 respectively.

We have

$$\sum_{j=1}^{2} P_{rj} = P_{r1} + P_{r2} = p + q = 1$$

and

$$\sum_{j=1}^{2} P_{mj} = P_{m1} + P_{m2} = iq + ip = i(1-p) + ip$$
$$= i - ip + ip = i = i(2-1) = i(N-1)$$

Where *N* is the number of random vectors or outcomes which is equal to 2 for a Bernoulli distribution.

The complex random vector corresponding to the random outcome x_1 is:

$$z_1 = P_{r1} + P_{m1} = p + i(1-p) = p + iq$$

The complex random vector corresponding to the random outcome x_2 is:

$$z_2 = P_{r2} + P_{m2} = q + i(1-q) = q + ip$$

The resultant complex random vector is defined as follows:

$$Z = \sum_{j=1}^{2} z_j = z_1 + z_2 = \sum_{j=1}^{2} P_{rj} + \sum_{j=1}^{2} P_{mj}$$
$$= (p + iq) + (q + ip) = (p + q) + i(p + q)$$
$$= 1 + i = 1 + i(2 - 1)$$
$$\Rightarrow Z = 1 + i(N - 1)$$

The probability Pc_1 in the complex plane $\mathcal{C} = \mathcal{R} + \mathcal{M}$ which corresponds to the complex random vector z_1 is computed as follows:

$$|z_1|^2 = P_{r1}^2 + (P_{m1} / i)^2 = p^2 + q^2$$
$$Chf_1 = -2P_{r1}P_{m1} / i = -2pq$$
$$\Rightarrow Pc_1^2 = |z_1|^2 - Chf_1$$
$$= p^2 + q^2 + 2pq = (p + q)^2 = 1^2 = 1$$
$$\Rightarrow Pc_1 = 1$$

This is coherent with the three novel complementary axioms defined for the *CPP*.

Similarly, Pc_2 corresponding to z_2 is:

$$|z_2|^2 = P_{r2}^2 + (P_{m2} / i)^2 = q^2 + p^2$$
$$Chf_2 = -2P_{r2}P_{m2} / i = -2qp$$
$$\Rightarrow Pc_2^2 = |z_2|^2 - Chf_2$$
$$= q^2 + p^2 + 2qp = (q + p)^2 = 1^2 = 1$$
$$\Rightarrow Pc_2 = 1$$

The probability Pc in the complex plane $\mathcal{C}$ which corresponds to the resultant complex random vector $Z = 1 + i$ is computed as follows:

$$|Z|^2 = \left(\sum_{j=1}^{2} P_{rj} \right)^2 + \left(\sum_{j=1}^{2} P_{mj} / i \right)^2 = 1^2 + 1^2 = 2$$

$$Chf = -2 \sum_{j=1}^{2} P_{rj} \sum_{j=1}^{2} P_{mj} / i = -2(1)(1) = -2$$

Let $s^2 = |Z|^2 - Chf = 2 + 2 = 4 \Rightarrow s = 2$

$$\Rightarrow Pc^2 = \frac{s^2}{N^2} = \frac{|Z|^2 - Chf}{N^2} = \frac{|Z|^2}{N^2} - \frac{Chf}{N^2} = \frac{4}{2^2} = \frac{4}{4} = 1$$

$$\Rightarrow Pc = \frac{s}{N} = \frac{2}{2} = 1$$

Where s is an intermediary quantity used in our computation of Pc.

Pc is the probability corresponding to the resultant complex random vector *Z* in the probability universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ and is also equal to 1. Actually, *Z* represents both z_1 and z_2 that means the whole distribution of random vectors of the general Bernoulli distribution in the complex plane $\mathcal{C}$ and its probability *Pc* is computed in the same way as Pc_1 and Pc_2.

By analogy, for the case of one random vector z_j we have:

$$Pc_j^2 = |z_j|^2 - Chf_j \quad \text{with} \quad (N = 1).$$

In general, for the vector *Z* we have:

$$Pc^2 = \frac{|Z|^2}{N^2} - \frac{Chf}{N^2}; \quad (N \geq 1)$$

Where the degree of our knowledge of the whole distribution is equal to $DOK_Z = \frac{|Z|^2}{N^2}$, its relative chaotic factor is $Chf_Z = \frac{Chf}{N^2}$, and its relative magnitude of the chaotic factor is $MChf_Z = |Chf_Z|$.

Notice, if *N* = 1 in the previous formula, then:

$$Pc^2 = \frac{|Z|^2}{N^2} - \frac{Chf}{N^2} = \frac{|Z|^2}{1^2} - \frac{Chf}{1^2} = |Z|^2 - Chf = |z_j|^2 - Chf_j = Pc_j^2$$

which is coherent with the calculations already done.

To illustrate the concept of the resultant complex random vector *Z*, I will use the following graph (Figure 15).

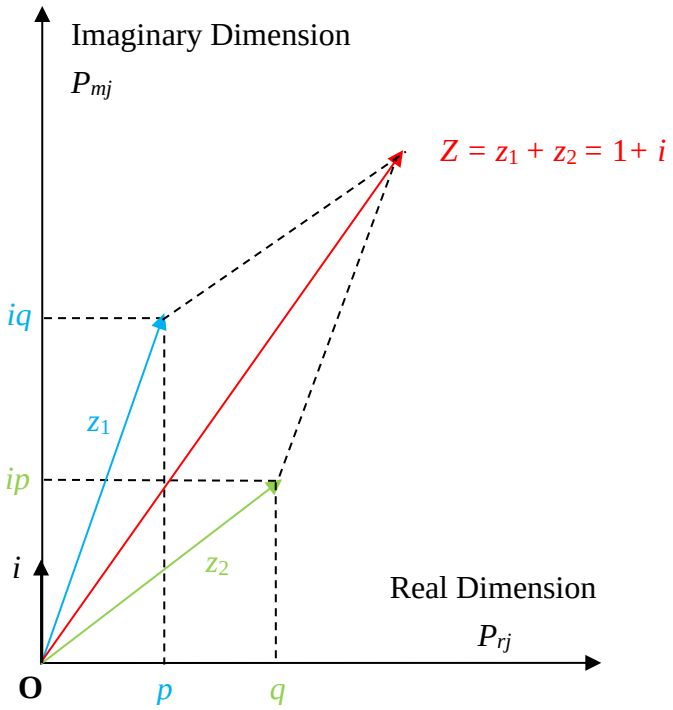


Figure 15. The resultant complex random vector $Z = z_1 + z_2$ for a general Bernoulli distribution in the complex probability plane $\mathcal{C}$

VI-2- The General Case: A Discrete Distribution with N Equiprobable Random Vectors (A Uniform Distribution U with N Random Outputs)

As a general case, let us consider then this discrete probability distribution with N equiprobable random vectors which is a discrete uniform probability distribution U with N outputs (Table 2):

Outcome	x_j	x_1	x_2	...	x_N
In $\mathcal{R}$	P_{rj}	$P_{r1}=\frac{1}{N}$	$P_{r2}=\frac{1}{N}$	...	$P_{rN}=\frac{1}{N}$
In $\mathcal{M}$	P_{mj}	$P_{m1}=i\left(1-\frac{1}{N}\right)$	$P_{m2}=i\left(1-\frac{1}{N}\right)$	...	$P_{mN}=i\left(1-\frac{1}{N}\right)$
In $\mathcal{C}=\mathcal{R}+\mathcal{M}$	z_j	$z_1=P_{r1}+P_{m1}$	$z_2=P_{r2}+P_{m2}$	...	$z_N=P_{rN}+P_{mN}$

Table 2. A discrete uniform distribution with N equiprobable random vectors in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$

We have here in $\mathcal{C}=\mathcal{R}+\mathcal{M}$:

$$z_j=P_{rj}+P_{mj},\quad \forall j:\ 1\le j\le N,$$

$$\text{and } z_1=z_2=\ldots=z_N=\frac{1}{N}+\frac{i(N-1)}{N}$$

$$\Rightarrow Z_U=\sum_{j=1}^{N}z_j=z_1+z_2+\ldots+z_N=Nz_j=N\left(\frac{1}{N}+\frac{i(N-1)}{N}\right)=1+i(N-1)$$

Moreover, we can notice that: $|z_1|=|z_2|=\cdots=|z_N|$, hence,

$$|Z_U|=|z_1+z_2+\ldots+z_N|=N|z_1|=N|z_2|=\ldots=N|z_N|$$

$$\Rightarrow |Z_U|^2=N^2|z_j|^2=N^2\left(\frac{1}{N^2}+\frac{(N-1)^2}{N^2}\right)=1+(N-1)^2,\ \text{where } 1\le j\le N;$$

And

$$Chf=N^2\times Chf_j=-2\times P_{rj}\times(P_{mj}/i)\times N^2=-2N^2\times\left(\frac{1}{N}\right)\left(\frac{N-1}{N}\right)=-2(1)(N-1)=-2(N-1)$$

$$\Rightarrow s^2=|Z_U|^2-Chf=1+(N-1)^2+2(N-1)=[1+(N-1)]^2=N^2$$

$$\Rightarrow Pc_U{}^2=\frac{s^2}{N^2}=\frac{N^2}{N^2}=1$$

$$=\frac{|Z_U|^2}{N^2}-\frac{Chf}{N^2}=\frac{1+(N-1)^2}{N^2}-\frac{-2(N-1)}{N^2}=\frac{1+(N-1)^2+2(N-1)}{N^2}=\frac{[1+(N-1)]^2}{N^2}=\frac{N^2}{N^2}=1$$

$$\Rightarrow Pc_U=1$$

Where s is an intermediary quantity used in our computation of Pc_U.

Therefore, the degree of our knowledge corresponding to the resultant complex vector Z_U representing the whole uniform distribution is:

$$DOK_{Z_U} = \frac{|Z_U|^2}{N^2} = \frac{1+(N-1)^2}{N^2},$$

and its relative chaotic factor is:

$$Chf_{Z_U} = \frac{Chf}{N^2} = -\frac{2(N-1)}{N^2},$$

Similarly, its relative magnitude of the chaotic factor is:

$$MChf_{Z_U} = |Chf_{Z_U}| = \left|\frac{Chf}{N^2}\right| = \left|-\frac{2(N-1)}{N^2}\right| = \frac{2(N-1)}{N^2}.$$

Thus, we can verify that we have always:

$$Pc_U^2 = \frac{|Z_U|^2}{N^2} - \frac{Chf}{N^2} = DOK_{Z_U} - Chf_{Z_U} = DOK_{Z_U} + MChf_{Z_U} = 1 \Leftrightarrow Pc_U = 1$$

What is important here is that we can notice the following fact. Take for example:

$$N = 2 \Rightarrow DOK_{Z_U} = \frac{1+(2-1)^2}{2^2} = 0.5 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(2-1)}{2^2} = -0.5$$

$$N = 4 \Rightarrow DOK_{Z_U} = \frac{1+(4-1)^2}{4^2} = 0.625 \geq 0.5 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(4-1)}{4^2} = -0.375 \geq -0.5$$

$$N = 5 \Rightarrow DOK_{Z_U} = \frac{1+(5-1)^2}{5^2} = 0.68 \geq 0.625 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(5-1)}{5^2} = -0.32 \geq -0.375$$

$$N = 10 \Rightarrow DOK_{Z_U} = \frac{1+(10-1)^2}{10^2} = 0.82 \geq 0.68 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(10-1)}{10^2} = -0.18 \geq -0.32$$

$$N = 100 \Rightarrow DOK_{Z_U} = \frac{1+(100-1)^2}{100^2} = 0.9802 \geq 0.82 \quad \text{and} \quad Chf_{Z_U} = \frac{-2(100-1)}{100^2} = -0.0198 \geq -0.18$$

$$N = 1000 \Rightarrow DOK_{Z_U} = \frac{1+(1000-1)^2}{1000^2} = 0.998002 \geq 0.9802 \quad \text{and}$$

$$Chf_{Z_U} = \frac{-2(1000-1)}{1000^2} = -0.001998 \geq -0.0198$$

We can deduce mathematically using calculus that:

$$\lim_{N\to+\infty} \frac{|Z_U|^2}{N^2} = \lim_{N\to+\infty} DOK_{Z_U} = \lim_{N\to+\infty} \frac{1+(N-1)^2}{N^2} = 1,$$

$$\text{and } \lim_{N\to+\infty} \frac{Chf}{N^2} = \lim_{N\to+\infty} Chf_{Z_U} = \lim_{N\to+\infty} -\frac{2(N-1)}{N^2} = 0.$$

From the above, we can also deduce this conclusion:

As much as N increases, as much as the degree of our knowledge in $\mathcal{R}$ corresponding to the resultant complex vector is perfect and absolute, that means, it is equal to one, and as much as the chaotic factor that prevents us from foretelling exactly and totally the outcome of the stochastic phenomenon in $\mathcal{R}$ approaches zero. Mathematically we state that: If N tends to infinity then the degree of our knowledge in $\mathcal{R}$ tends to one and the chaotic factor tends to zero.

VI-3- The Convergence of Monte Carlo Methods using *Z* and *CPP*

Subsequently, if $\lim_{N\to+\infty} Chf_{Z_U} = 0$ then $\lim_{N\to+\infty} Chf_{MC} = 0$ (the chaotic factor of Monte Carlo methods) provided that:

1) The Monte Carlo algorithm used to solve the stochastic process or integral is correct
2) The integral that we want to solve using Monte Carlo methods is convergent

Therefore:

$$1)\ \Rightarrow \lim_{N\to+\infty} Chf_{MC} = \lim_{N\to+\infty} -2P_r P_m / i = 0 \Rightarrow \begin{cases} P_r \to 0 \\ \text{OR} \\ P_m / i \to 0 \end{cases} \Rightarrow \begin{cases} P_r \to 0 \\ \text{OR} \\ P_r = 1 - P_m / i \to 1 - 0 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} P_{rob}(\text{convergence}) \to 0 \\ \text{OR} \\ P_{rob}(\text{convergence}) \to 1 \end{cases}$$

that means either the simulation has not started yet ($P_{rob}(\text{convergence}) = 0$) or the Monte Carlo algorithm result or output has converged to the exact result ($P_{rob}(\text{convergence}) \to 1$) since $Chf_{MC} = 0$ in only two places which are $N = 0$ and $N \to +\infty$.

$$2)\ \text{And} \lim_{N\to+\infty} Chf_{MC} = \begin{cases} \lim_{N\to+\infty} -2\left(\frac{R_A}{R_E}\right)\left(1-\frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ \lim_{N\to+\infty} -2\left(2-\frac{R_A}{R_E}\right)\left(\frac{R_A}{R_E}-1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = 0$$

$$\Rightarrow \begin{cases} \frac{R_A}{R_E} \to 0 \ \text{ OR } \ 1-\frac{R_A}{R_E} \to 0 \\ \text{OR} \\ 2-\frac{R_A}{R_E} \to 0 \ \text{ OR } \ \frac{R_A}{R_E}-1 \to 0 \end{cases} \Rightarrow \begin{cases} R_A \to 0 \ \text{ OR } \ R_A \to R_E \\ \text{OR} \\ R_A \to 2R_E \ \text{ OR } \ R_A \to R_E \end{cases}$$

that means either:

- the simulation has not started yet ($R_A = 0$ or $R_A = 2R_E$) since at this instant the percent relative error is maximum and is equal to 100%,
- or the Monte Carlo algorithm output has converged to the exact result ($R_A \to R_E$) since at this instant the percent relative error is minimum and is equal to 0%,

this is due to the fact that $Chf_{MC} = 0$ in only two places which are $N = 0$ and $N \to +\infty$.

Moreover, the speed of the convergence of Monte Carlo methods depends on:

1) The algorithm used

2) The integrand function of the original integral that we want to evaluate ($f(x)$ or in general $f(x_1, x_2, \ldots, x_n)$) since in Monte Carlo methods:

$$\int_a^b f(x)dx \cong \frac{b-a}{N}\sum_{j=1}^{N} f(x_j)$$

Or in general:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\ldots\int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_n).dx_1dx_2\ldots dx_n \cong \frac{[(b_1-a_1)\times(b_2-a_2)\times\ldots\times(b_n-a_n)]}{N}\sum_{j=1}^{N} f(x_{1j}, x_{2j}, \ldots, x_{nj})$$

3) The random numbers generator that provides the integrand function with random inputs for the Monte Carlo methods. In the current research work we have used one specific uniform random numbers generator although many others exist in literature.

Furthermore, for $N=1 \Rightarrow \frac{|Z|^2}{N^2} = DOK_{Z_U} = \frac{1+(1-1)^2}{1^2} = 1 \Rightarrow DOK_{MC} = 1$ (the DOK of Monte Carlo methods)

$$\text{And } \frac{Chf}{N^2} = Chf_{Z_U} = \frac{-2(1-1)}{1^2} = 0 \Rightarrow Chf_{MC} = 0$$

This means that we have a random experiment with only one outcome or vector, hence, either $P_r = 1$ (always converging) or $P_r = 0$ (always diverging), that means we have respectively either a sure event or an impossible event in $\mathcal{R}$. Consequently, we have surely the degree of our knowledge is equal to one (perfect experiment knowledge) and the chaotic factor is equal to zero (no chaos) since the experiment is either certain (that means we have used a deterministic algorithm so the stochastic Monte Carlo methods are replaced by deterministic methods that do not use random numbers like the classical and ordinary methods of numerical integration) or impossible (an incorrect or divergent algorithm or integral), which is absolutely logical.

Consequently, we have proved here the law of large numbers (already discussed in the published papers [37-90]) as well as the convergence of Monte Carlo methods using *CPP*. The following figures (Figures 16 and 17) show the convergence of Chf_{Z_U} to 0 and of DOK_{Z_U} to 1 as functions of the uniform samples number *N* (Number of inputs/outputs).

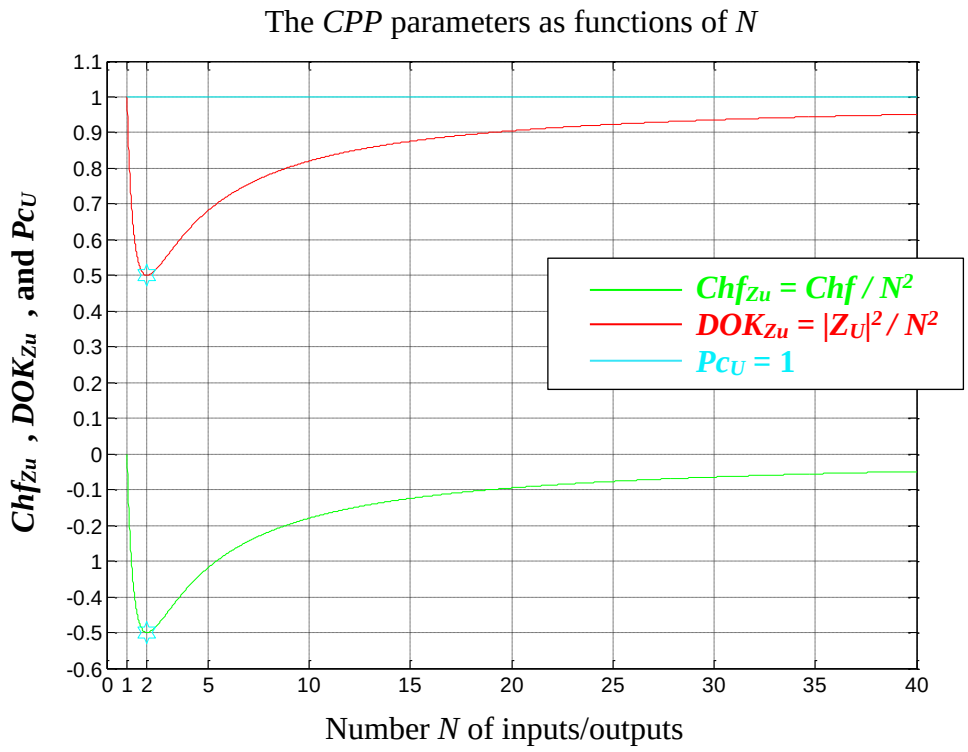


Figure 16. Chf_{Z_U}, DOK_{Z_U}, and Pc_U, as functions of N in 2D

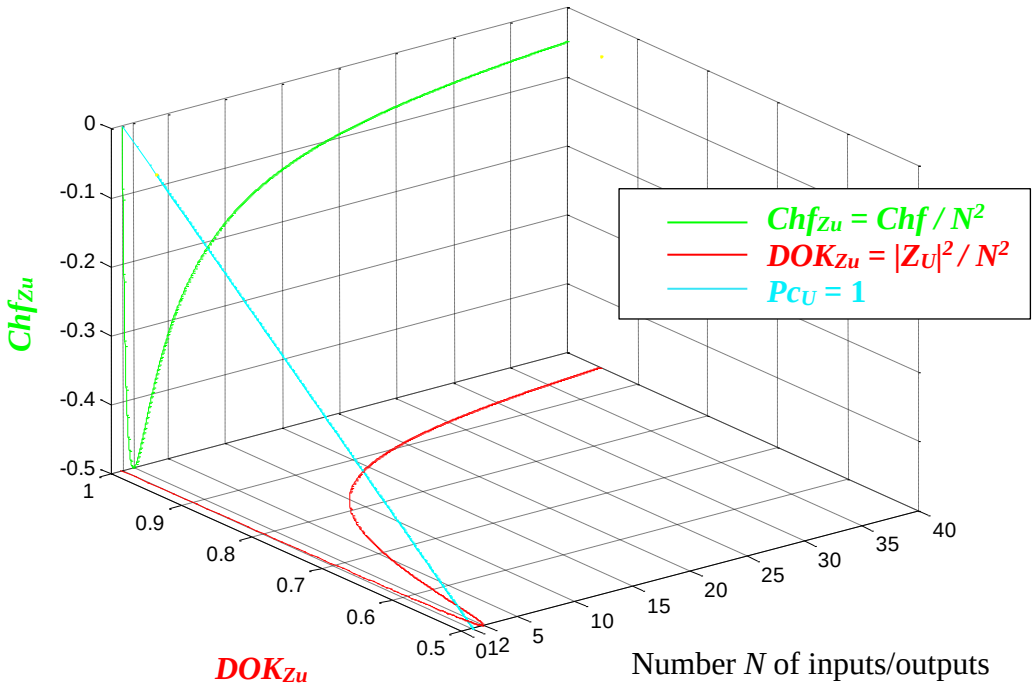


Figure 17. Chf_{Z_U}, DOK_{Z_U}, and Pc_U, as functions of N in 3D

VII- The Evaluation of the New Paradigm Parameters

We can deduce from what has been elaborated previously the following:

The real convergence probability: $P_r(N) = 1 - \left| \frac{R_E - R_A(N)}{R_E} \right|$

We have $0 \le N \le N_C$ where $N = 0$ corresponds to the instant before the beginning of the random experiment when $R_A(N=0) = 0$ or $= 2R_E$, and $N = N_C$ (iterations number needed for the method convergence) corresponds to the instant at the end of the random experiments and Monte Carlo methods when $R_A(N = N_C) \to R_E$.

The imaginary divergence probability: $P_m(N) = i \left| \frac{R_E - R_A(N)}{R_E} \right|$

The real complementary divergence probability: $P_m(N)/i = \left| \frac{R_E - R_A(N)}{R_E} \right|$

The complex probability and random vector:

$$Z(N) = P_r(N) + P_m(N) = \left[1 - \left| \frac{R_E - R_A(N)}{R_E} \right| \right] + i \left| \frac{R_E - R_A(N)}{R_E} \right|$$

The Degree of Our Knowledge:

$$DOK(N) = |Z(N)|^2 = P_r^2(N) + [P_m(N)/i]^2 = \left[1 - \left| \frac{R_E - R_A(N)}{R_E} \right| \right]^2 + \left[\left| \frac{R_E - R_A(N)}{R_E} \right| \right]^2$$

$$= 1 + 2iP_r(N)P_m(N) = 1 - 2P_r(N)[1 - P_r(N)] = 1 - 2P_r(N) + 2P_r^2(N)$$

$$= 1 - 2\left| \frac{R_E - R_A(N)}{R_E} \right| + 2\left[\frac{R_E - R_A(N)}{R_E} \right]^2$$

The Chaotic Factor:

$$Chf(N) = 2iP_r(N)P_m(N) = -2P_r(N)[1 - P_r(N)] = -2P_r(N) + 2P_r^2(N)$$

$$= -2\left| \frac{R_E - R_A(N)}{R_E} \right| + 2\left[\frac{R_E - R_A(N)}{R_E} \right]^2$$

$Chf(N)$ is null when $P_r(N) = P_r(0) = 0$ and when $P_r(N) = P_r(N_C) = 1$.

The Magnitude of the Chaotic Factor *MChf*:

$$MChf(N) = |Chf(N)| = -2iP_r(N)P_m(N) = 2P_r(N)[1 - P_r(N)] = 2P_r(N) - 2P_r^2(N)$$

$$= 2\left| \frac{R_E - R_A(N)}{R_E} \right| - 2\left[\frac{R_E - R_A(N)}{R_E} \right]^2$$

$MChf(N)$ is null when $P_r(N) = P_r(0) = 0$ and when $P_r(N) = P_r(N_C) = 1$.

At any iteration number *N*: $0 \le \forall N \le N_C$, the probability expressed in the complex probability set $\mathcal{C}$ is the following:

$$\begin{aligned} Pc^2(N) &= [P_r(N) + P_m(N)/i]^2 = |Z(N)|^2 - 2iP_r(N)P_m(N) \\ &= DOK(N) - Chf(N) \\ &= DOK(N) + MChf(N) \\ &= 1 \end{aligned}$$

then,

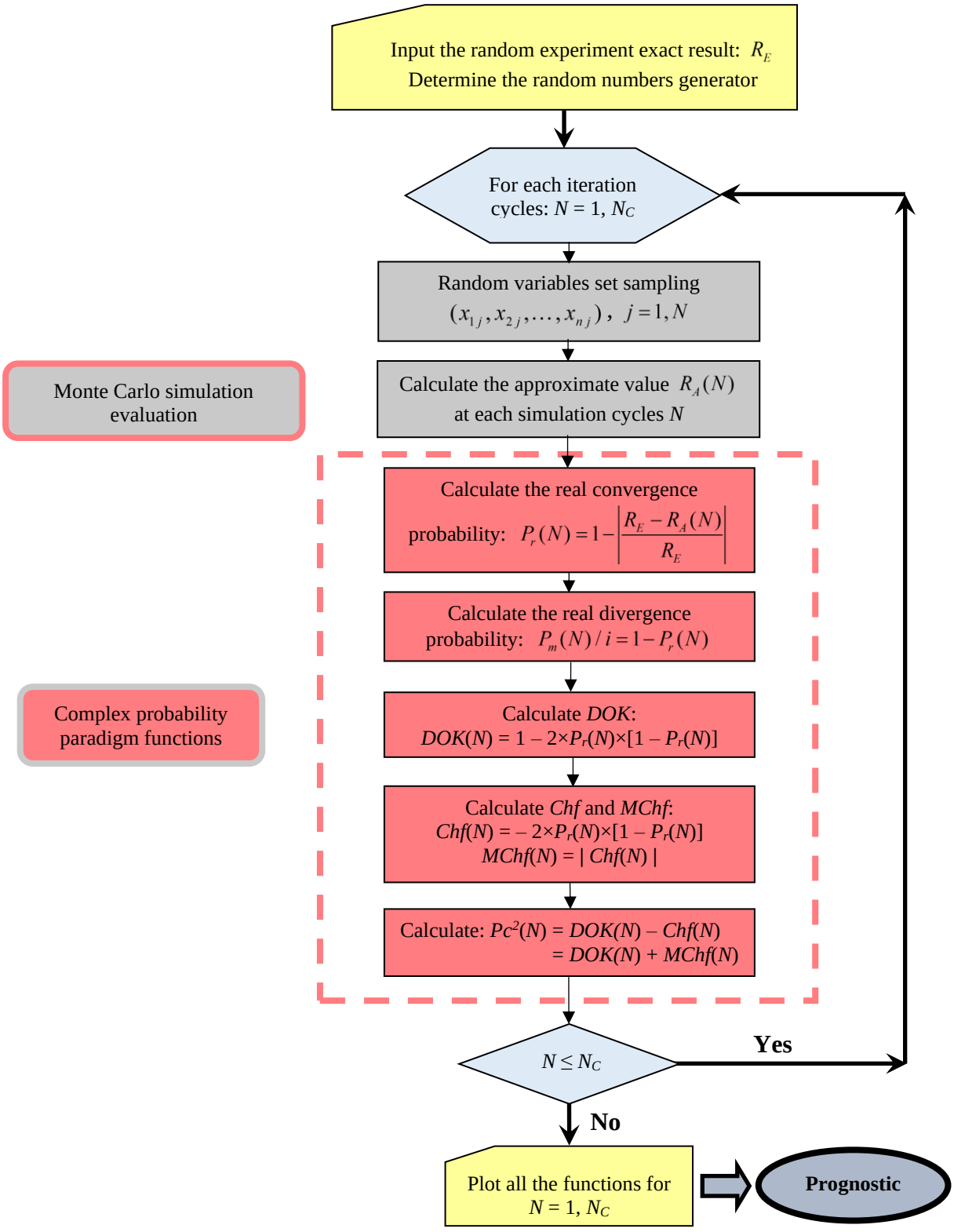
$$Pc^2(N) = [P_r(N) + P_m(N)/i]^2 = \{P_r(N) + [1 - P_r(N)]\}^2 = 1^2 \Leftrightarrow Pc = 1 \text{ always}$$

Hence, the prediction of the convergence probabilities of the stochastic Monte Carlo experiments in the set $\mathcal{C}$ is permanently certain.

Let us consider thereafter some stochastic experiments and some single and multidimensional integrals to simulate the Monte Carlo methods and to draw, to visualize, as well as to quantify all the *CPP* and prognostic parameters.

VIII- Flowchart of the Complex Probability and Monte Carlo Methods Prognostic Model

The following flowchart summarizes all the procedures of the proposed complex probability prognostic model:



IX- Simulation of the New Paradigm

Note that all the numerical values found in the simulations of the new paradigm for any iteration cycles *N* were computed using the MATLAB version 2024 software and compared to the values found by Microsoft Visual C++ programs (included in all section IX-2). In addition, the reader should take care of the rounding errors since all numerical values are represented by at most five significant digits and since we are using Monte Carlo methods of integration and simulation which give approximate results subject to random effects and fluctuations.

IX-1- The Continuous Random Case

IX-1-1- The First Simple Integral: A Linear Function

Let us consider the integral of the following linear function:

$\int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1^2}{2} - 0 = \frac{1}{2} = 0.5 \Leftrightarrow R_E = 0.5$ by the deterministic methods of calculus.

$\Leftrightarrow f(x) = x, \quad x \mapsto U(0,1)$

$\Leftrightarrow \int_0^1 x dx \cong \frac{1}{N}\sum_{j=1}^{N} x_j = R_A$ with $1 \le N \le N_C$ after applying Monte Carlo method.

Moreover, the four figures (Figures 18-21) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E = 0.5$ for *N* = 50, 100, 500, and $N = N_C = 100{,}000$ iterations. Therefore, we have:

$\lim_{N \to +\infty} P_r(N) = \lim_{N \to +\infty} \left\{1 - \left|\frac{R_E - R_A(N)}{R_E}\right|\right\} = 1 - \left|\frac{R_E - R_E}{R_E}\right| = 1 - 0 = 1$ which is equal to the convergence probability of Monte Carlo method as $N \to +\infty$.

Additionally, Figure 22 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters $(Chf, R_A, P_r, MChf, R_E, DOK, P_m / i, Pc)$ after applying it to this linear function.

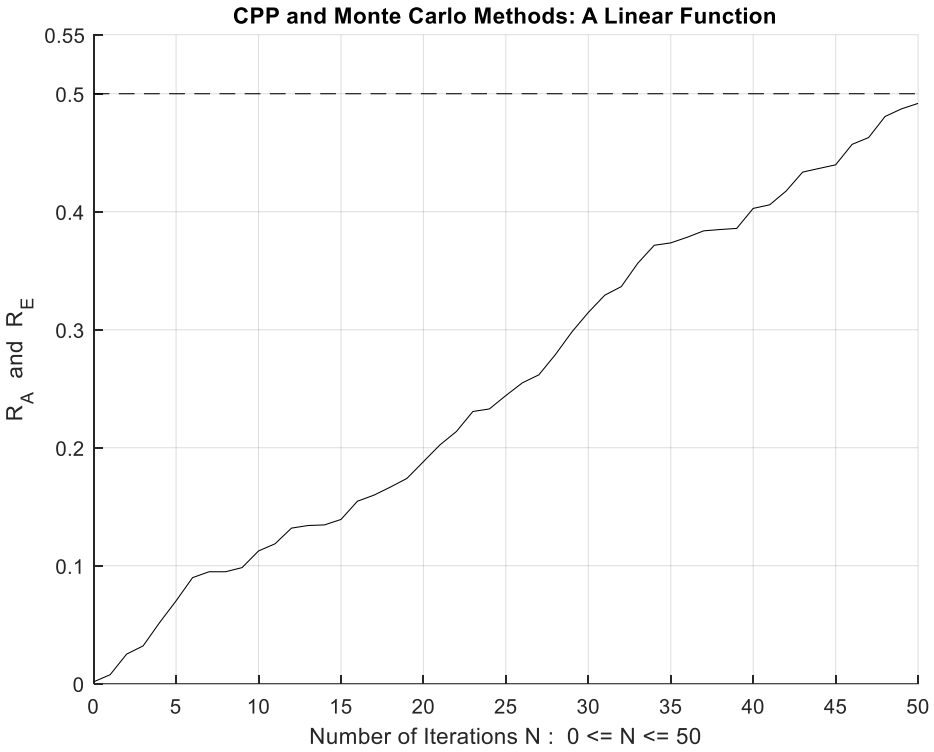


Figure 18. The increasing convergence of the Monte Carlo method up to $N = 50$ iterations.

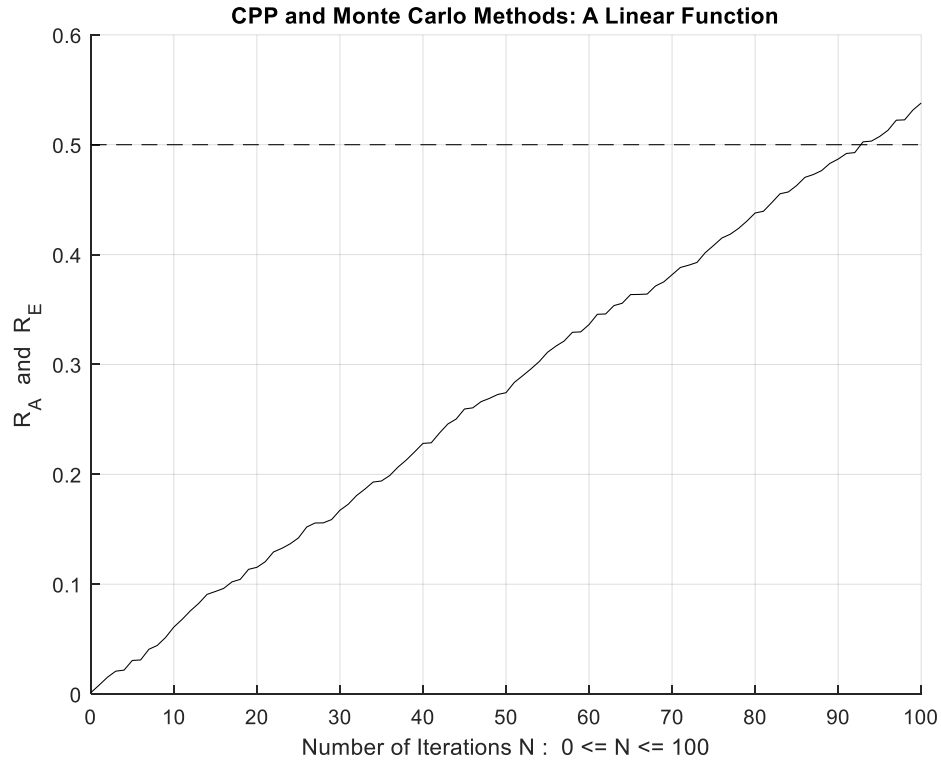


Figure 19. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

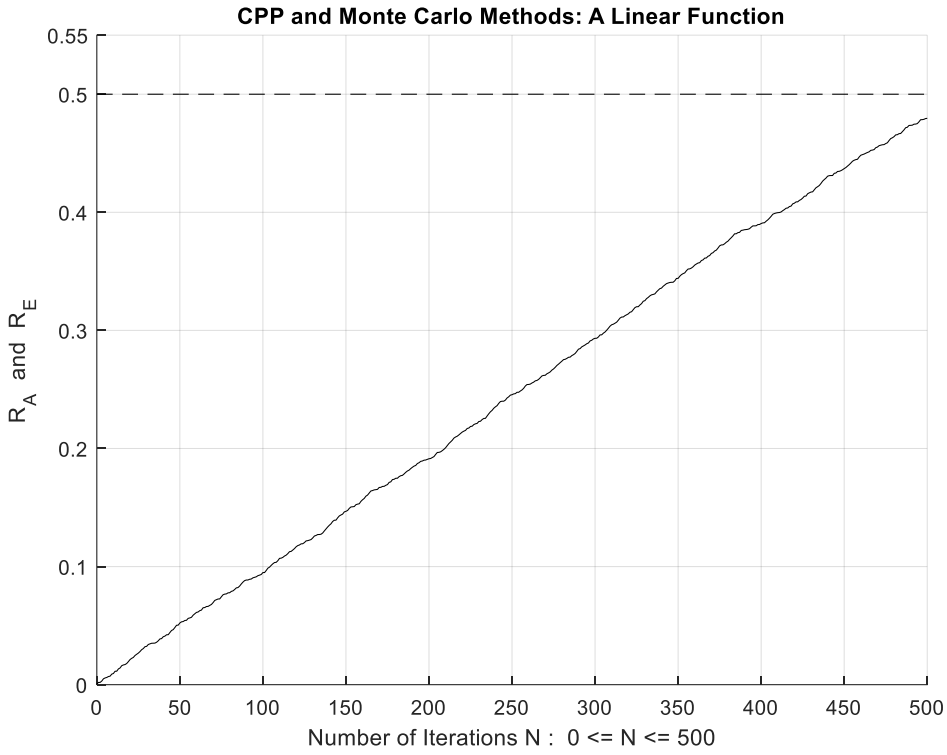


Figure 20. The increasing convergence of the Monte Carlo method up to N = 500 iterations.

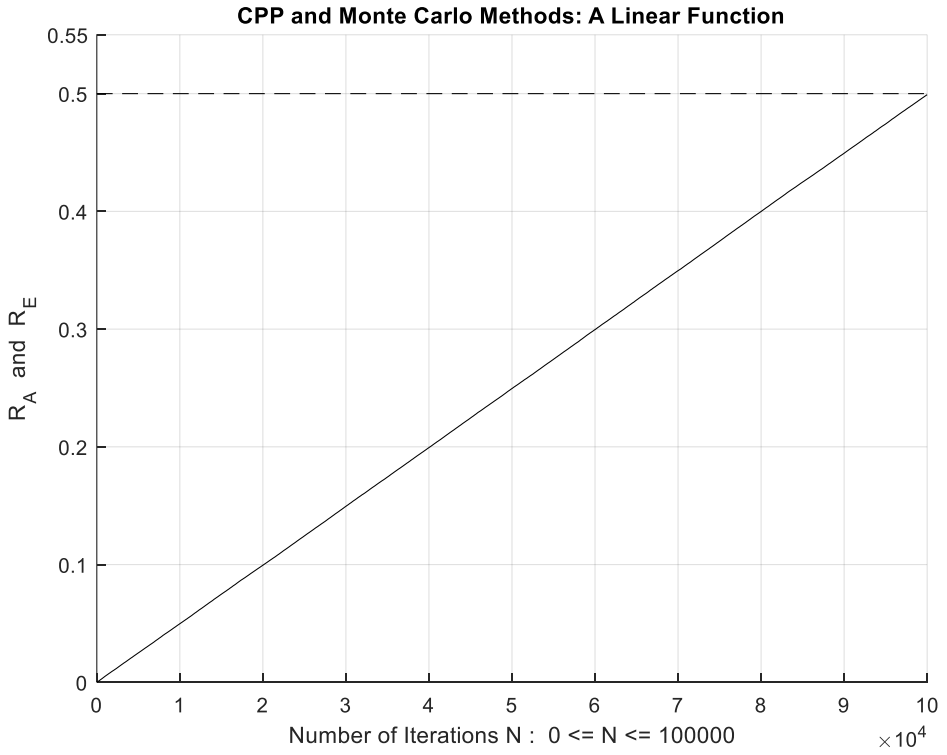


Figure 21. The increasing convergence of the Monte Carlo method up to N = 100,000 iterations.

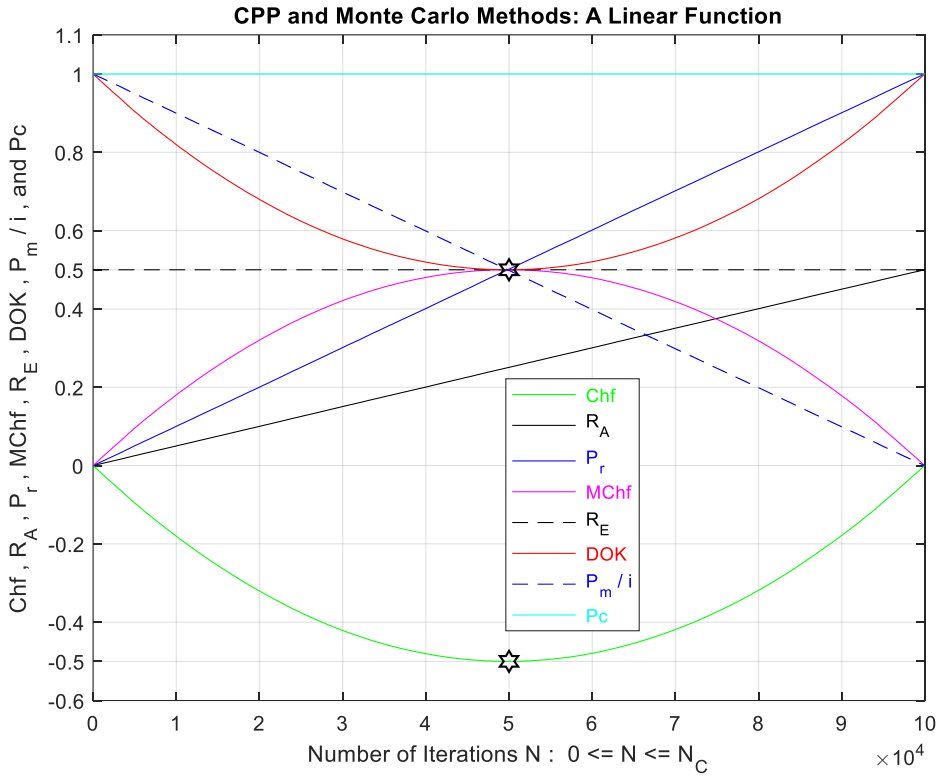


Figure 22. The *CPP* parameters and the Monte Carlo method for a linear function.

IX-1-2- The Second Simple Integral: A Parabolic Function

Let us consider the integral of the following parabolic function:

$\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - 0 = \frac{1}{3} = 0.3333333\ldots \Leftrightarrow R_E = 0.3333333\ldots$ by the deterministic methods of calculus.

$\Leftrightarrow f(x) = x^2, \quad x \mapsto U(0,1)$

$\Leftrightarrow \int_0^1 x^2 dx \cong \frac{1}{N}\sum_{j=1}^{N} x_j^2 = R_A$ with $1 \le N \le N_C$ after applying Monte Carlo method.

Moreover, the four figures (Figures 23-26) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E = 0.3333333\ldots$ for N = 50, 100, 500, and $N = N_C = 100{,}000$ iterations. Therefore, we have:

$\lim_{N\to+\infty} P_r(N) = \lim_{N\to+\infty}\left\{1 - \left|\frac{R_E - R_A(N)}{R_E}\right|\right\} = 1 - \left|\frac{R_E - R_E}{R_E}\right| = 1 - 0 = 1$ which is equal to the convergence probability of Monte Carlo method as $N \to +\infty$.

Additionally, Figure 27 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters $(Chf, R_A, P_r, MChf, R_E, DOK, P_m / i, Pc)$ after applying it to this parabolic function.

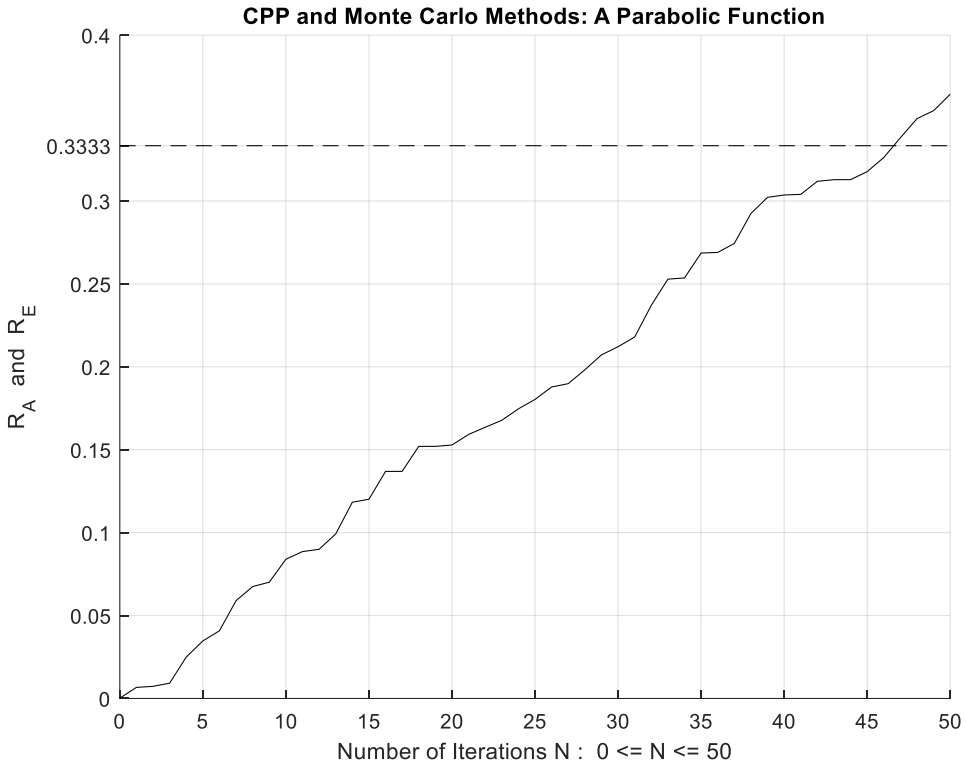


Figure 23. The increasing convergence of the Monte Carlo method up to $N = 50$ iterations.

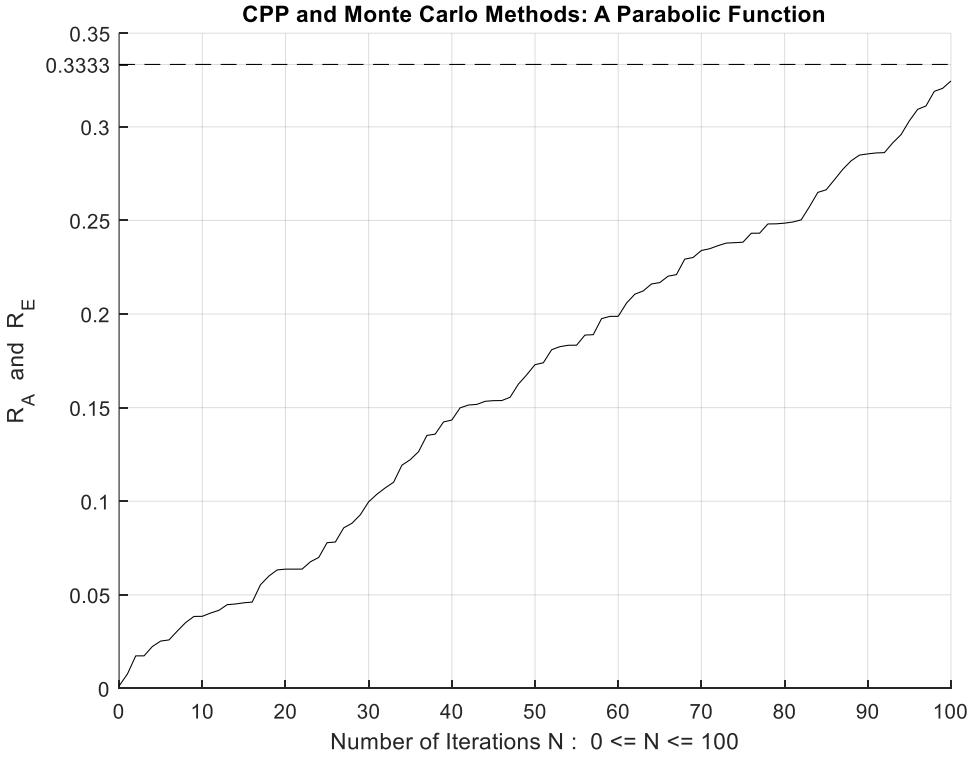


Figure 24. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

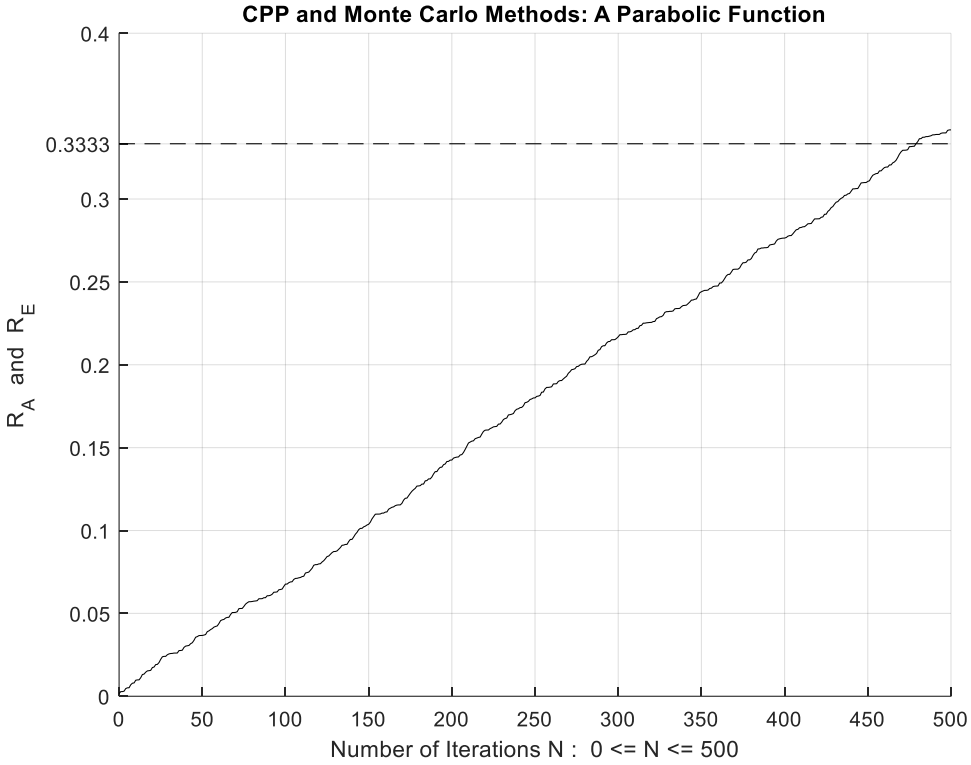


Figure 25. The increasing convergence of the Monte Carlo method up to N = 500 iterations.

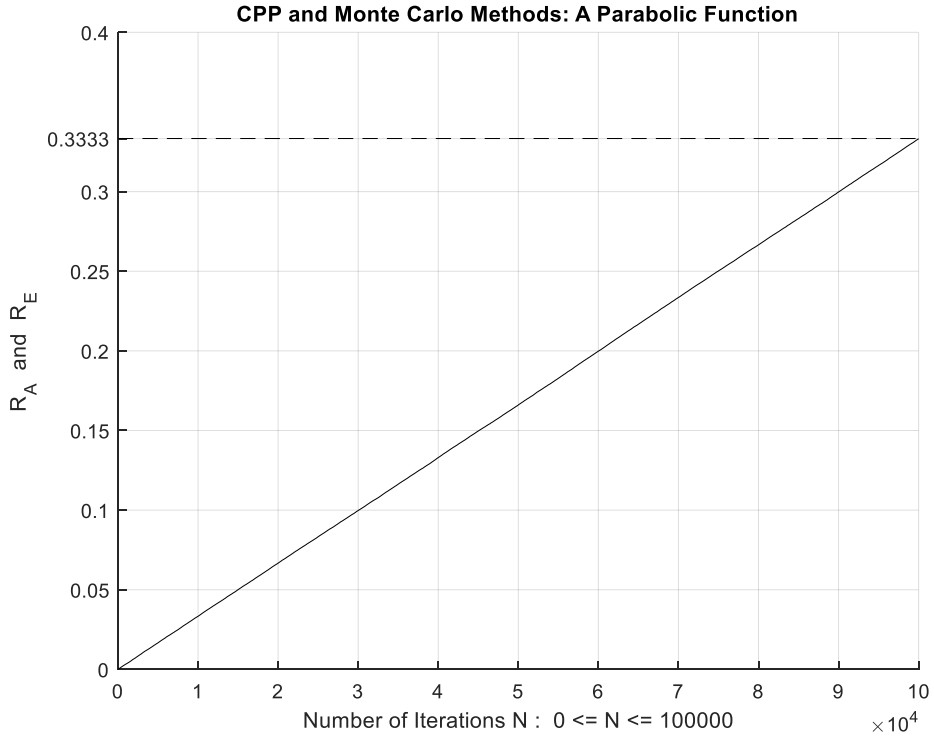


Figure 26. The increasing convergence of the Monte Carlo method up to N = 100,000 iterations.

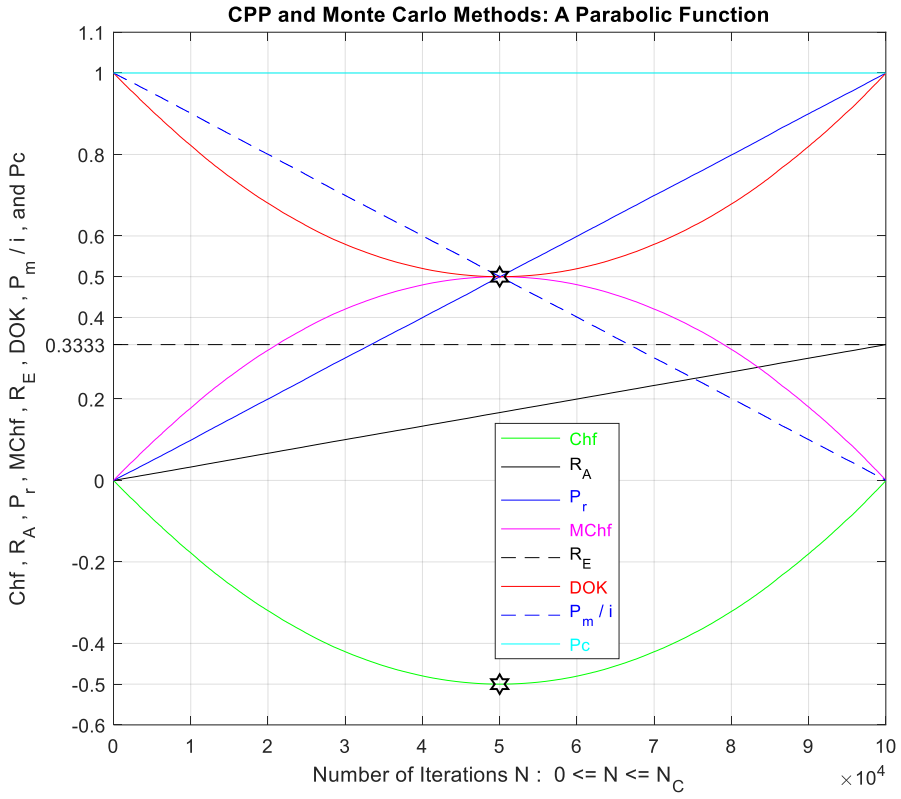


Figure 27. The *CPP* parameters and the Monte Carlo method for a parabolic function.

IX-1-3- The Third Simple Integral: A Cubic Function

Let us consider the integral of the following cubic function:

$\int_0^1 x^3 dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1^4}{4} - 0 = \frac{1}{4} = 0.25 \Leftrightarrow R_E = 0.25$ by the deterministic methods of calculus.

$\Leftrightarrow f(x) = x^3, \quad x \mapsto U(0,1)$

$\Leftrightarrow \int_0^1 x^3 dx \cong \frac{1}{N}\sum_{j=1}^{N} x_j^3 = R_A$ with $1 \le N \le N_C$ after applying Monte Carlo method.

Moreover, the four figures (Figures 28-31) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E = 0.25$ for N = 50, 100, 500, and $N = N_C = 100{,}000$ iterations. Therefore, we have:

$\lim_{N\to+\infty} P_r(N) = \lim_{N\to+\infty}\left\{1 - \left|\frac{R_E - R_A(N)}{R_E}\right|\right\} = 1 - \left|\frac{R_E - R_E}{R_E}\right| = 1 - 0 = 1$ which is equal to the convergence probability of Monte Carlo method as $N \to +\infty$.

Additionally, Figure 32 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters $(Chf, R_A, P_r, MChf, R_E, DOK, P_m / i, Pc)$ after applying it to this cubic function.

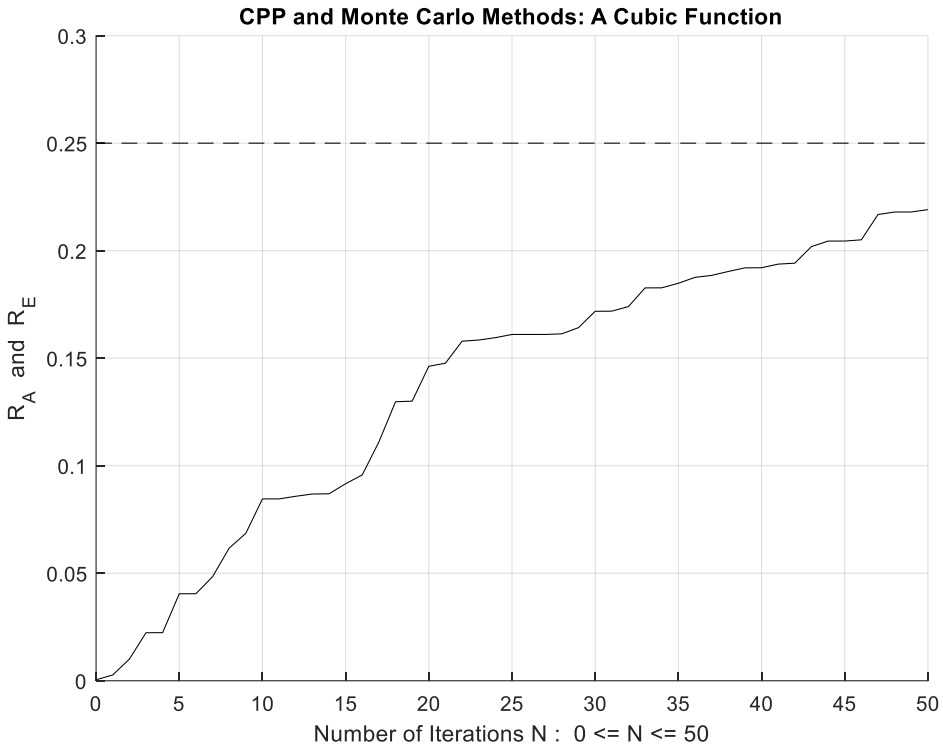


Figure 28. The increasing convergence of the Monte Carlo method up to $N = 50$ iterations.

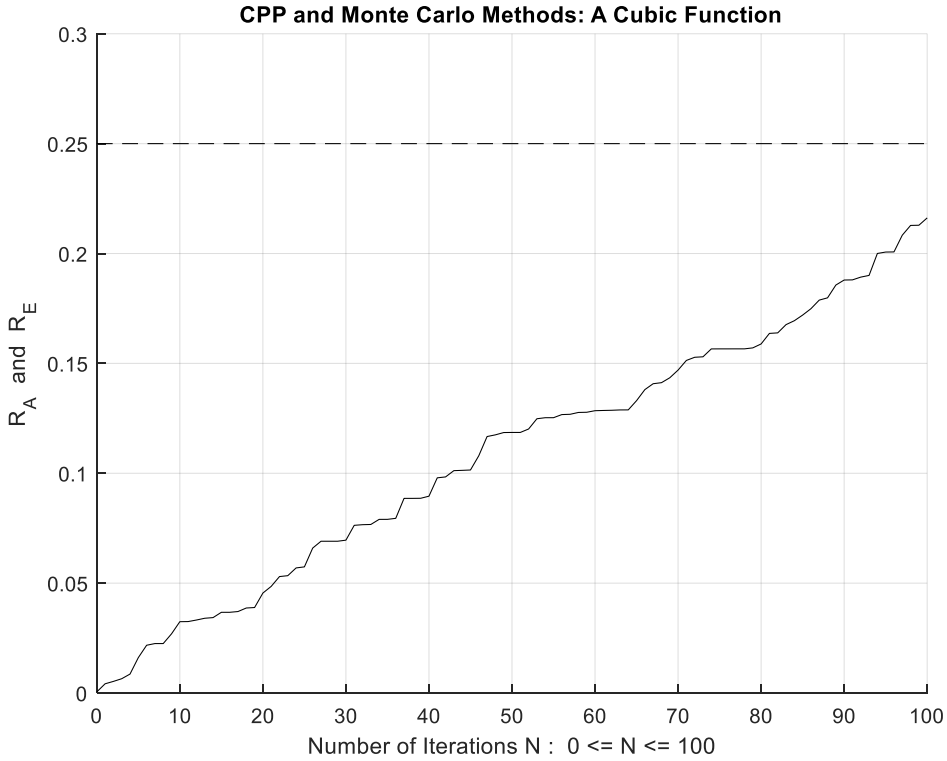


Figure 29. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

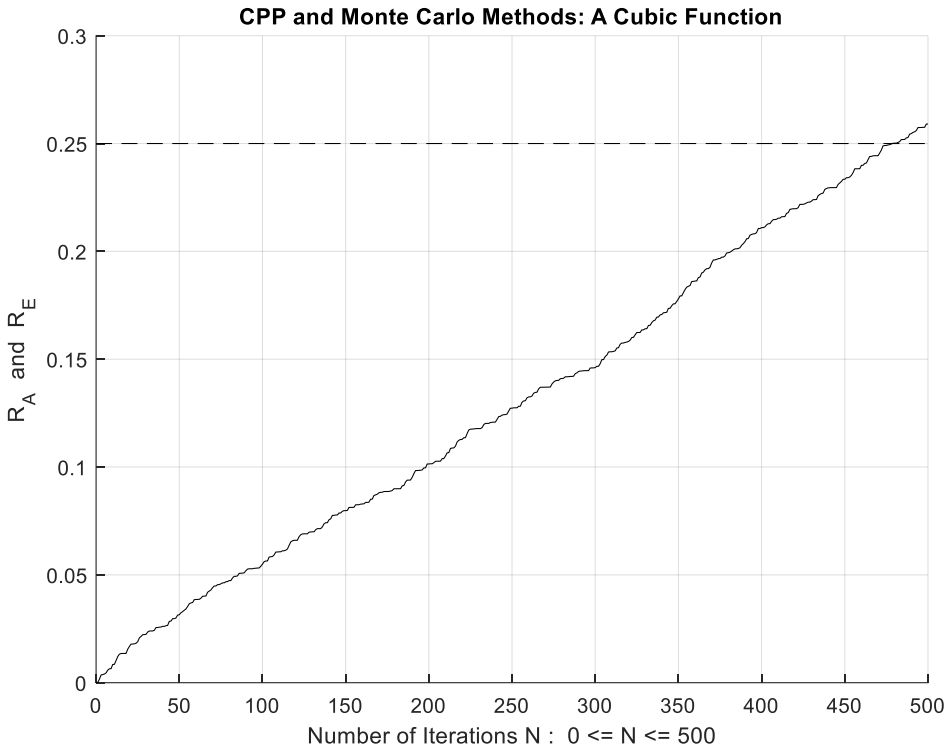


Figure 30. The increasing convergence of the Monte Carlo method up to N = 500 iterations.

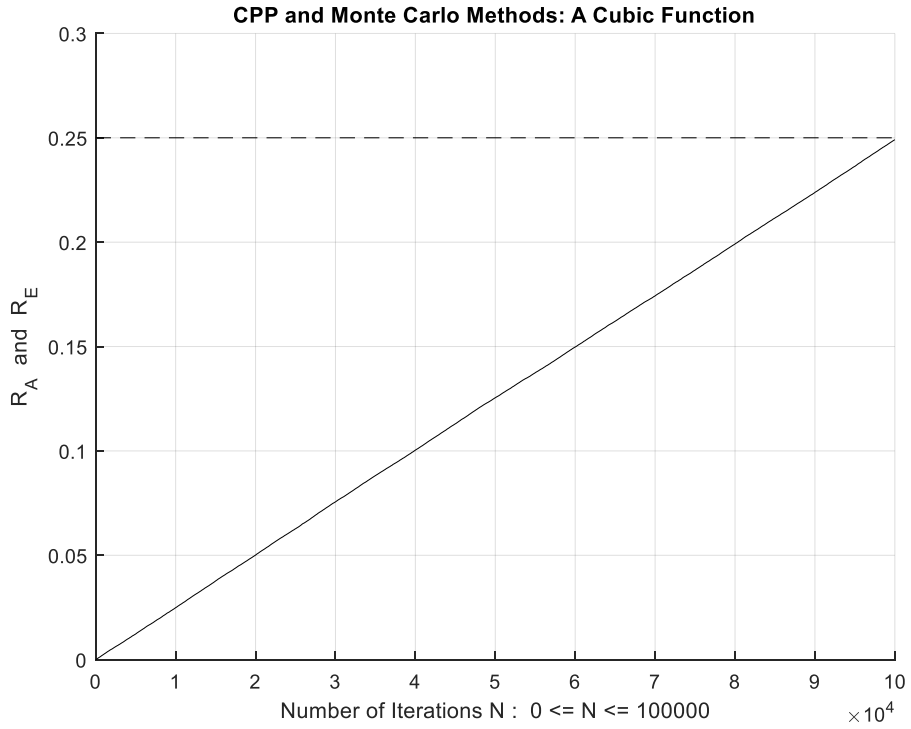


Figure 31. The increasing convergence of the Monte Carlo method up to N = 100,000 iterations.

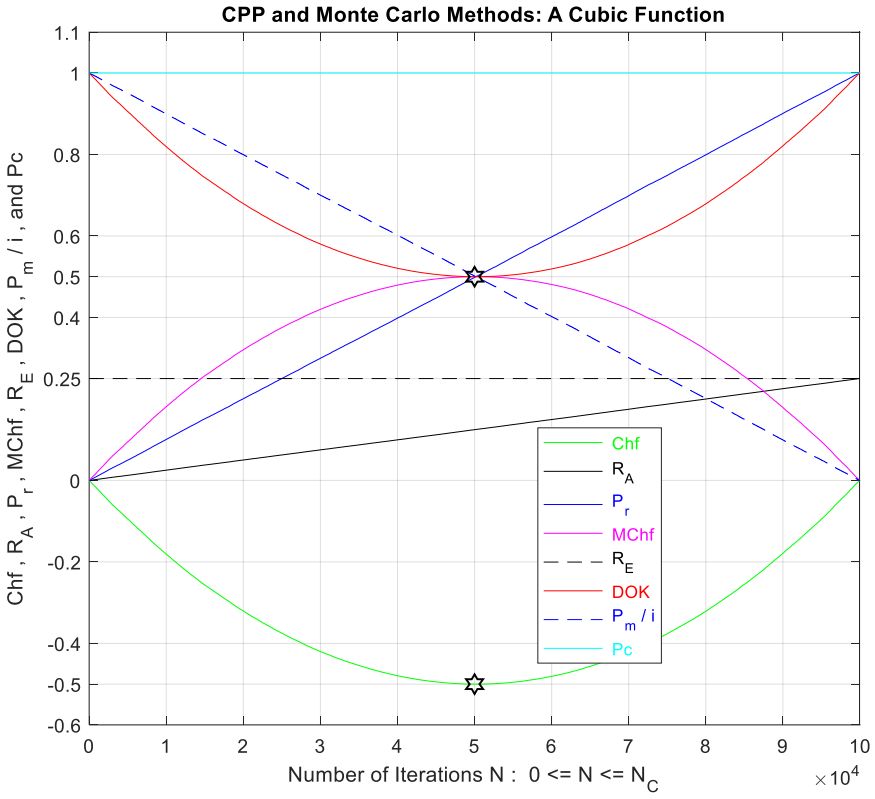


Figure 32. The *CPP* parameters and the Monte Carlo method for a cubic function.

IX-1-4- The Fourth Simple Integral: An Increasing Exponential Function

Let us consider the integral of the following increasing exponential function:

$$\int_0^1 \exp(x)dx = \left[\exp(x)\right]_0^1 = \exp(1)-\exp(0) = e-1 = 2.718281828\ldots-1 = 1.718281828\ldots$$

$\Leftrightarrow R_E = 1.718281828\ldots$ by the deterministic methods of calculus.

$\Leftrightarrow f(x) = \exp(x)$, $x \mapsto U(0,1)$

$\Leftrightarrow \int_0^1 \exp(x)dx \cong \frac{1}{N}\sum_{j=1}^{N}\exp(x_j) = R_A$ with $1 \le N \le N_C$ after applying Monte Carlo method.

Moreover, the four figures (Figures 33-36) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E = 1.718281828\ldots$ for N = 50, 100, 500, and $N = N_C = 100,000$ iterations. Therefore, we have:

$$\lim_{N\to+\infty} P_r(N) = \lim_{N\to+\infty}\left\{1-\left|\frac{R_E - R_A(N)}{R_E}\right|\right\} = 1-\left|\frac{R_E - R_E}{R_E}\right| = 1-0 = 1$$ which is equal to the convergence probability of Monte Carlo method as $N \to +\infty$.

Additionally, Figure 37 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters $(Chf, R_A, P_r, MChf, R_E, DOK, P_m/i, Pc)$ after applying it to this increasing exponential function.

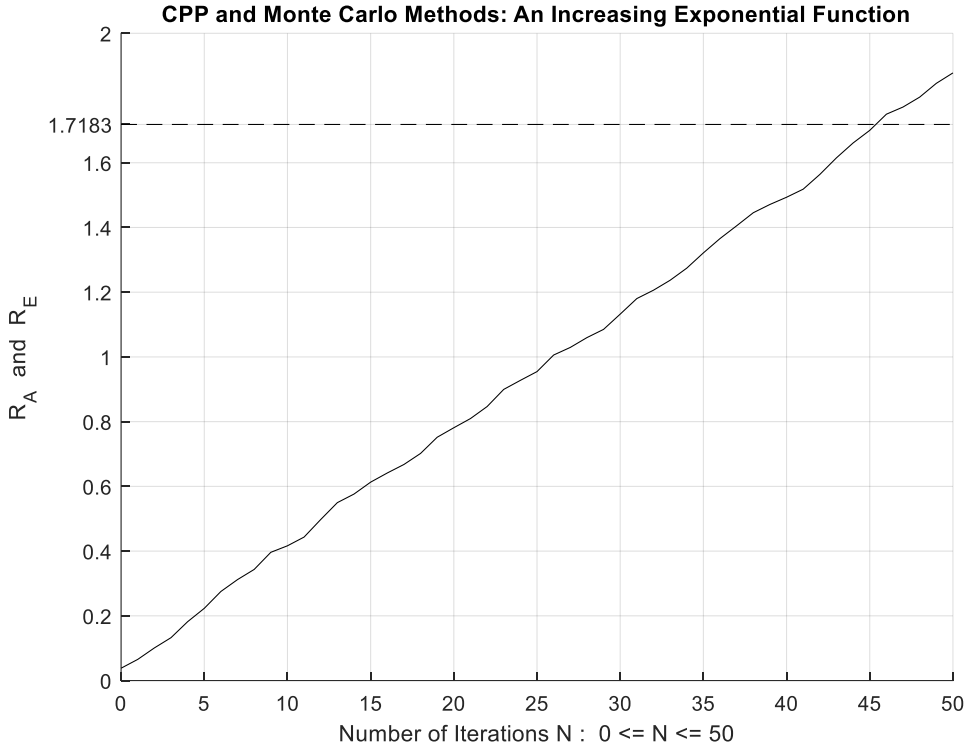


Figure 33. The increasing convergence of the Monte Carlo method up to $N = 50$ iterations.

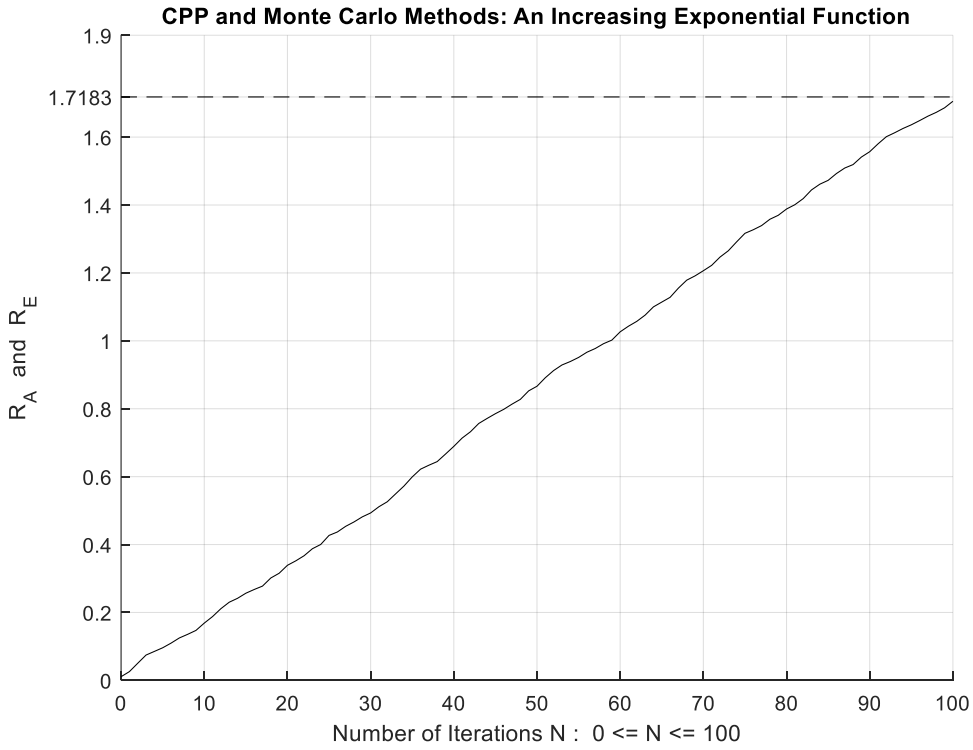


Figure 34. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

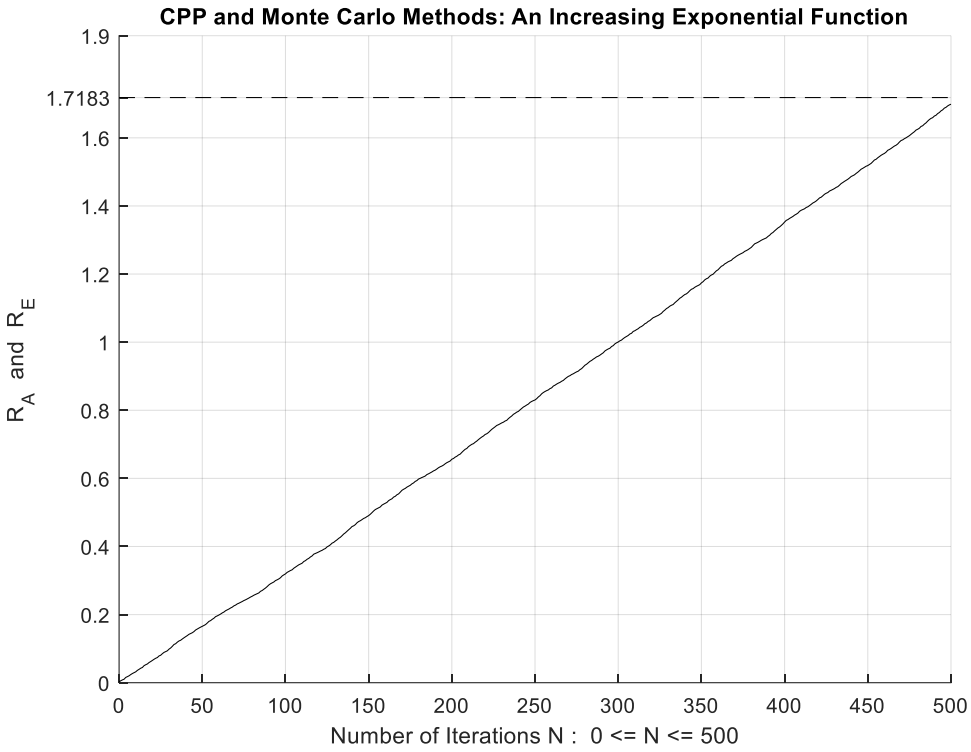


Figure 35. The increasing convergence of the Monte Carlo method up to N = 500 iterations.

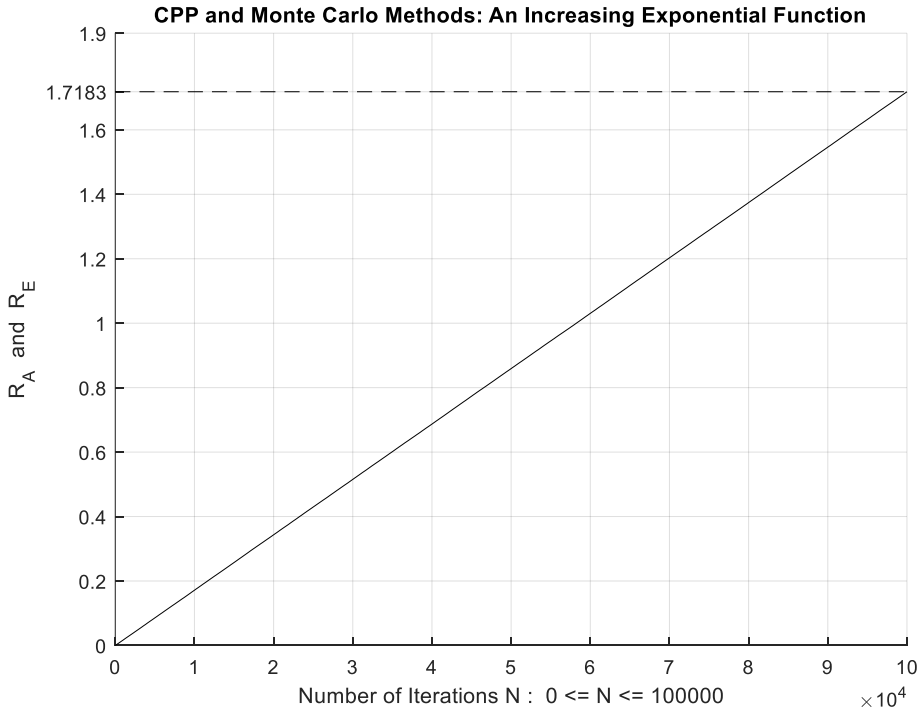


Figure 36. The increasing convergence of the Monte Carlo method up to N = 100,000 iterations.

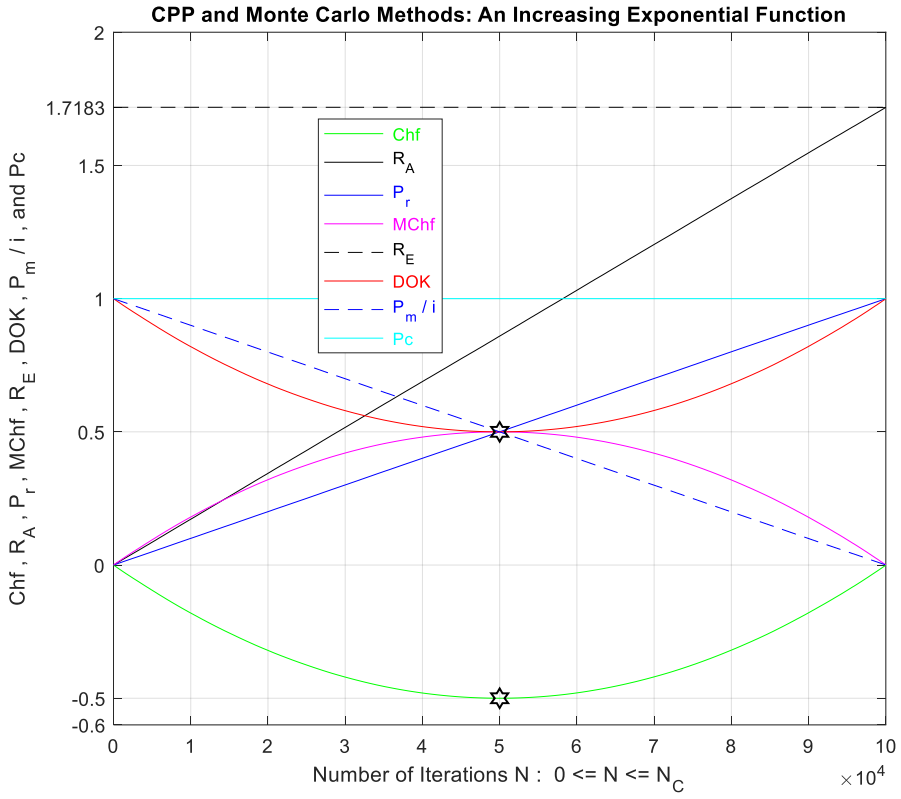


Figure 37. The *CPP* parameters and the Monte Carlo method for an increasing exponential function.

IX-1-5- The Fifth Simple Integral: A Decreasing Exponential Function

Let us consider the integral of the following decreasing exponential function:

$$\int_0^1 \exp(-x)dx = \left[-\exp(-x)\right]_0^1 = -\exp(-1)+\exp(0) = -0.367879441\ldots+1 = 0.632120558\ldots$$

$\Leftrightarrow R_E = 0.632120558\ldots$ by the deterministic methods of calculus.

$\Leftrightarrow f(x) = \exp(-x),\ \ x \mapsto U(0,1)$

$\Leftrightarrow \int_0^1 \exp(-x)dx \cong \frac{1}{N}\sum_{j=1}^{N}\exp(-x_j) = R_A$ with $1 \le N \le N_C$ after applying Monte Carlo method.

Moreover, the four figures (Figures 38-41) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E = 0.632120558\ldots$ for N = 50, 100, 500, and $N = N_C = 100{,}000$ iterations. Therefore, we have:

$\lim_{N\to+\infty} P_r(N) = \lim_{N\to+\infty}\left\{1-\left|\frac{R_E - R_A(N)}{R_E}\right|\right\} = 1-\left|\frac{R_E - R_E}{R_E}\right| = 1-0 = 1$ which is equal to the convergence probability of Monte Carlo method as $N \to +\infty$.

Additionally, Figure 42 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters $(Chf, R_A, P_r, MChf, R_E, DOK, P_m/i, Pc)$ after applying it to this decreasing exponential function.

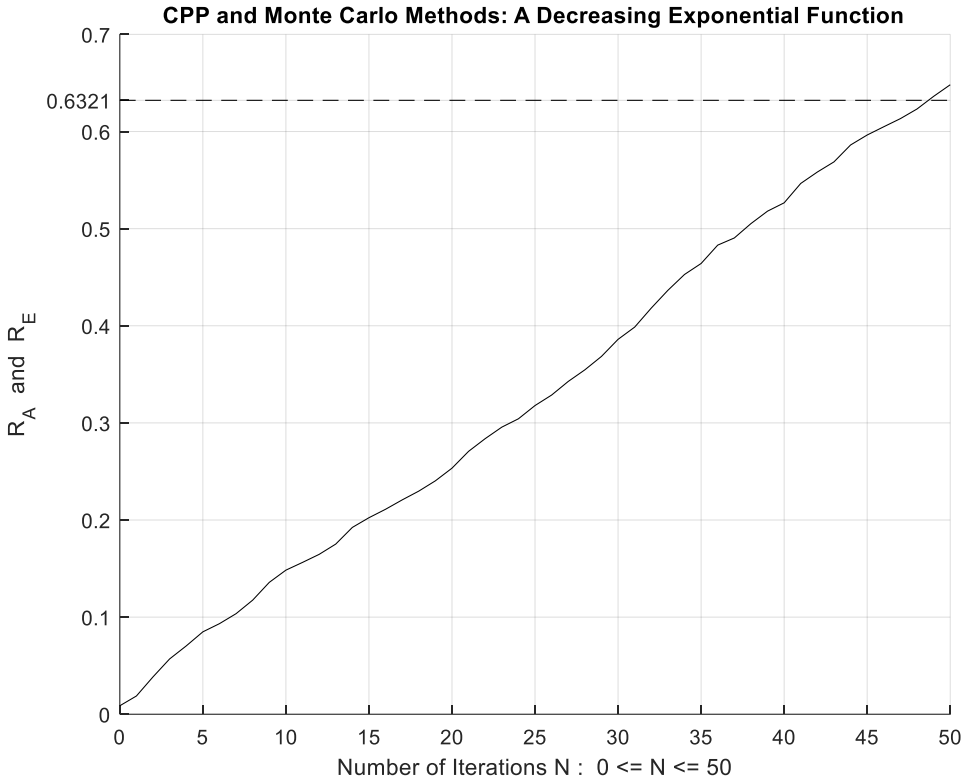


Figure 38. The increasing convergence of the Monte Carlo method up to $N = 50$ iterations.

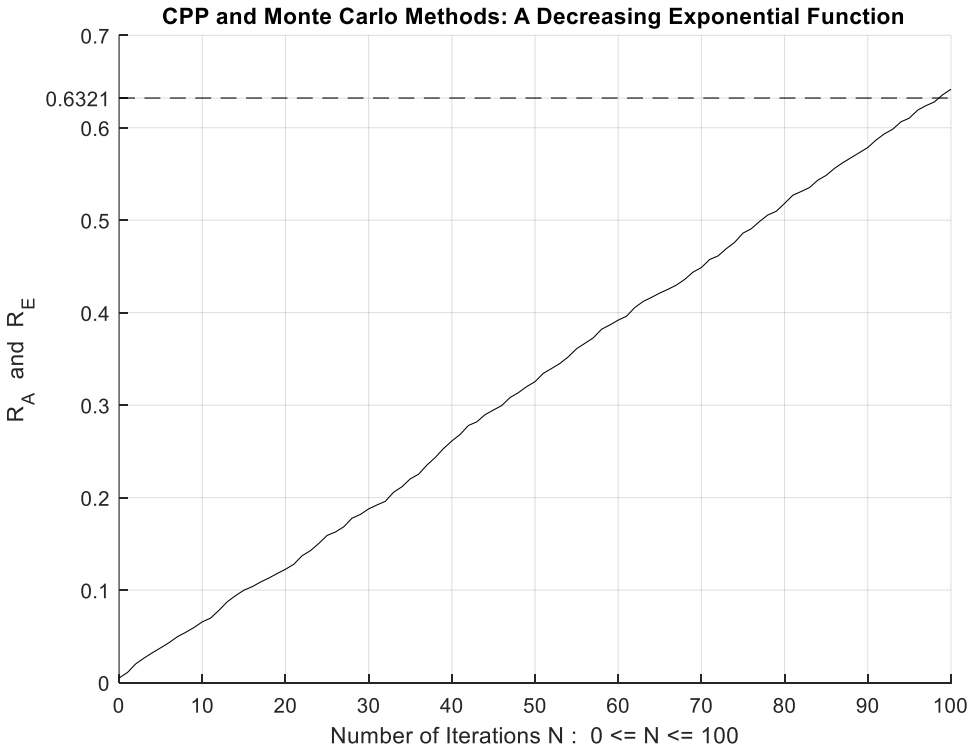


Figure 39. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

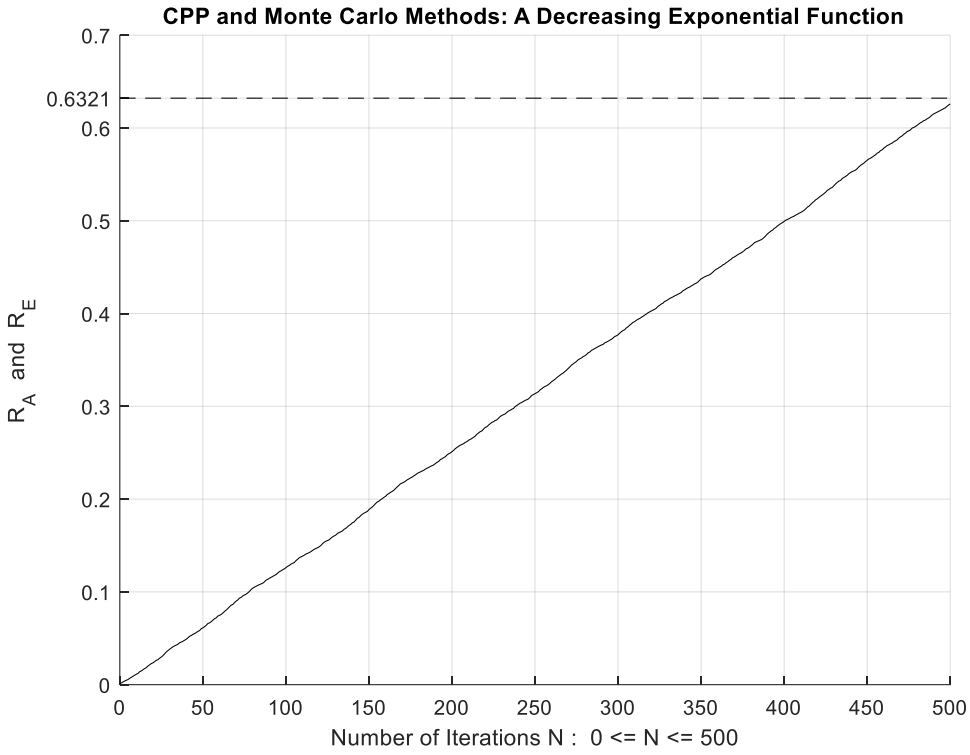


Figure 40. The increasing convergence of the Monte Carlo method up to N = 500 iterations.

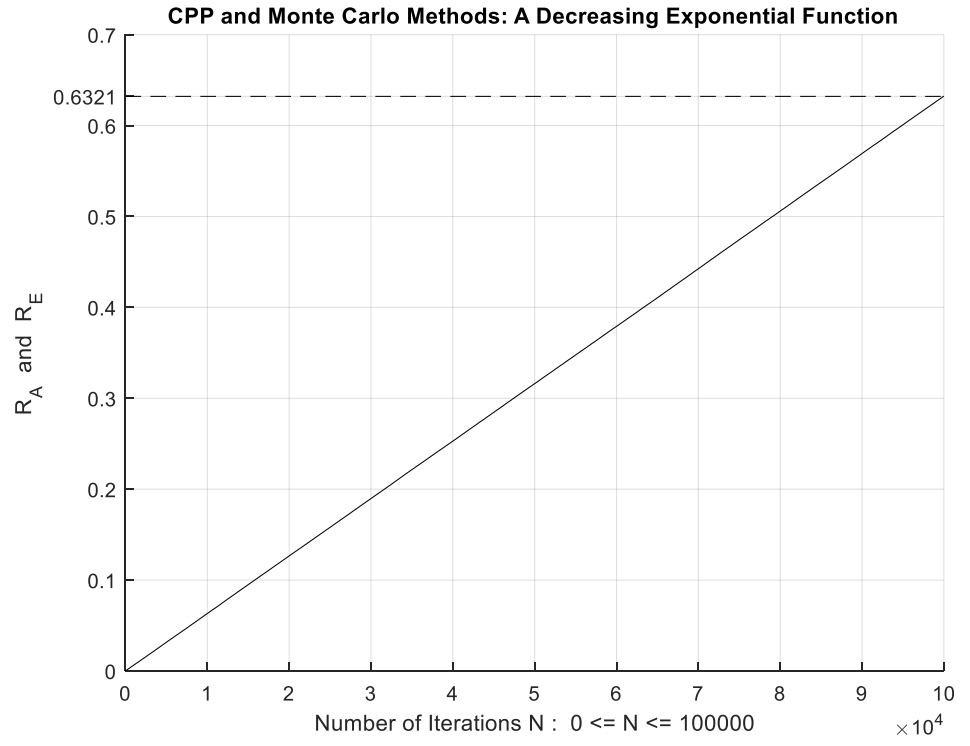


Figure 41. The increasing convergence of the Monte Carlo method up to N = 100,000 iterations.

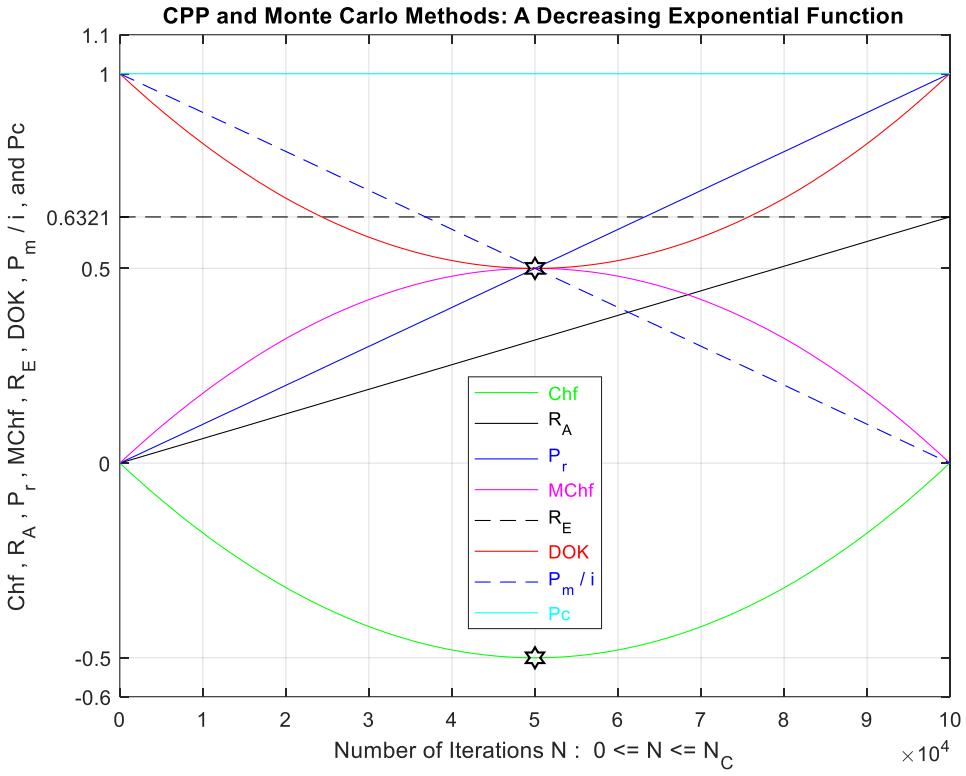


Figure 42. The *CPP* parameters and the Monte Carlo method for a decreasing exponential function.

IX-1-6- The Sixth Simple Integral: A Logarithmic Function

Let us consider the integral of the following logarithmic function:

$\int_1^2 Ln(x)dx = \left[xLn(x)-x\right]_1^2 = 2Ln2-1 = 0.386294361\ldots \Leftrightarrow R_E = 0.386294361\ldots$ by the deterministic methods of calculus.

$\Leftrightarrow f(x) = Ln(x),\quad x \mapsto U(1,2)$

$\Leftrightarrow \int_0^1 Ln(x)dx \cong \frac{1}{N}\sum_{j=1}^{N} Ln(x_j) = R_A$ with $1 \le N \le N_C$ after applying Monte Carlo method.

Moreover, the four figures (Figures 43-46) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E = 0.386294361\ldots$ for N = 50, 100, 500, and $N = N_C = 100,000$ iterations. Therefore, we have:

$\lim_{N\to+\infty} P_r(N) = \lim_{N\to+\infty}\left\{1-\left|\frac{R_E - R_A(N)}{R_E}\right|\right\} = 1-\left|\frac{R_E - R_E}{R_E}\right| = 1-0 = 1$ which is equal to the convergence probability of Monte Carlo method as $N \to +\infty$.

Additionally, Figure 47 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters $(Chf, R_A, P_r, MChf, R_E, DOK, P_m/i, Pc)$ after applying it to this logarithmic function.

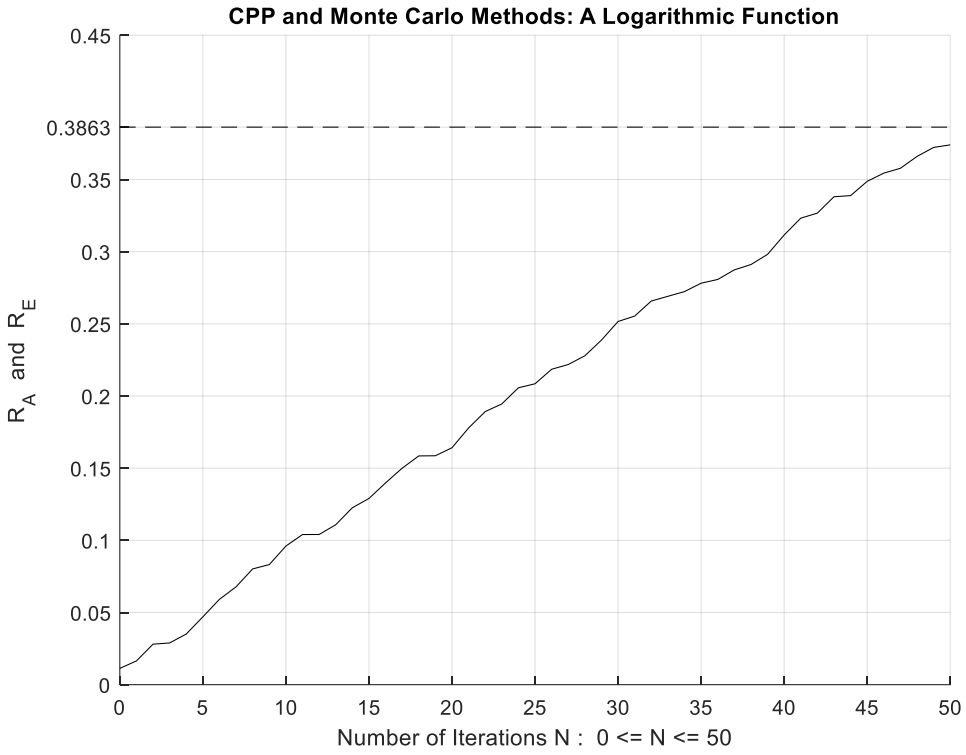


Figure 43. The increasing convergence of the Monte Carlo method up to $N = 50$ iterations.

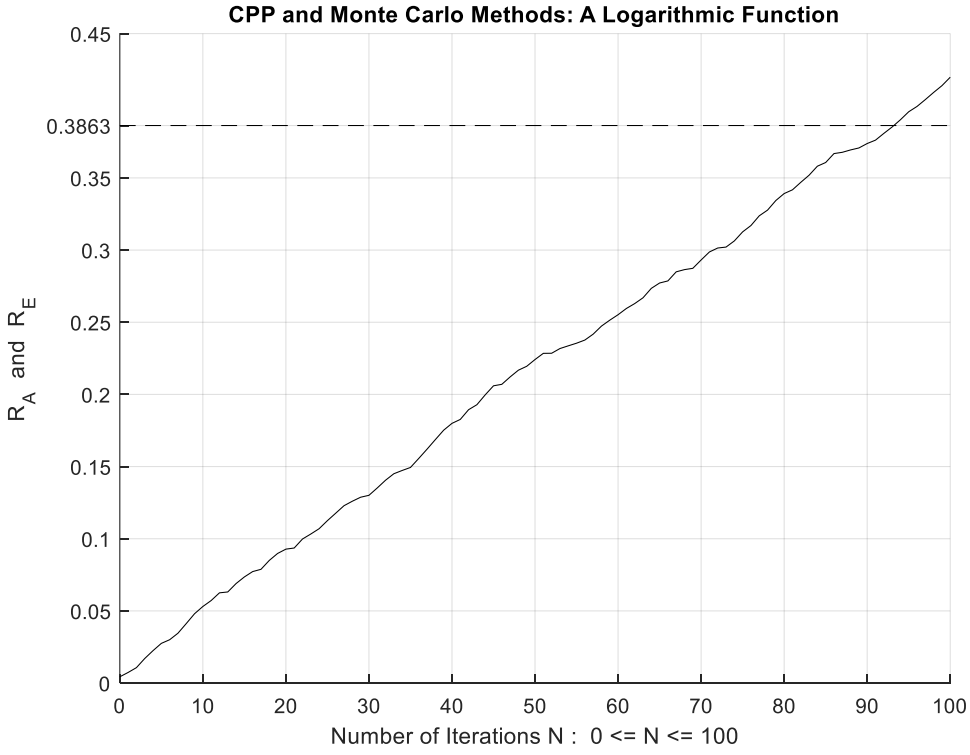


Figure 44. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

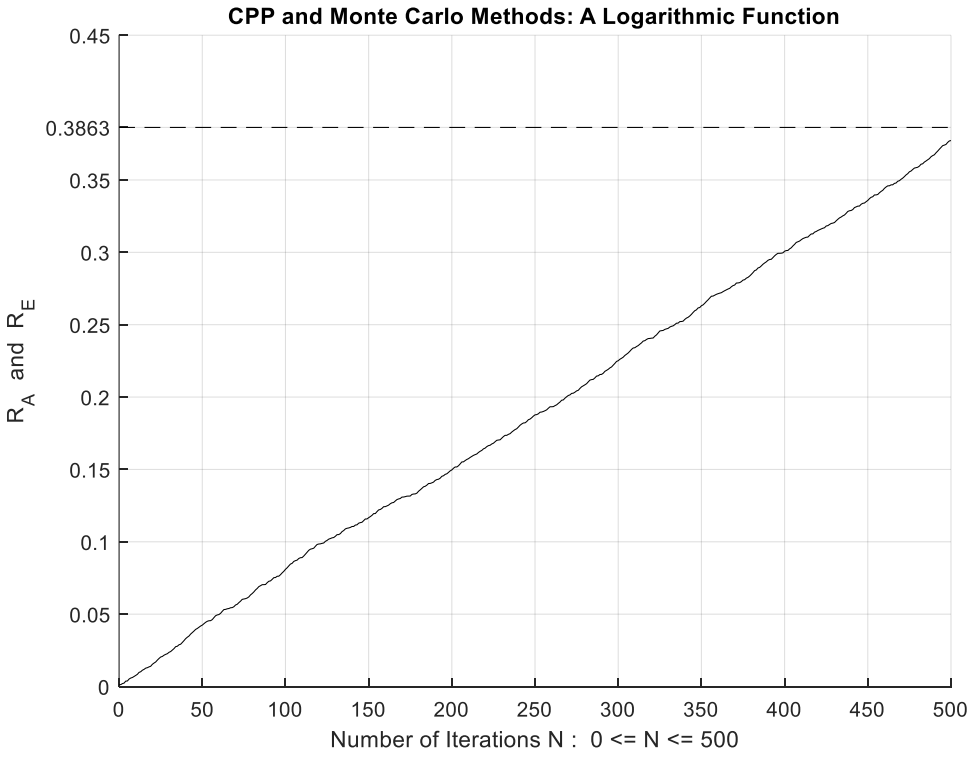


Figure 45. The increasing convergence of the Monte Carlo method up to N = 500 iterations.

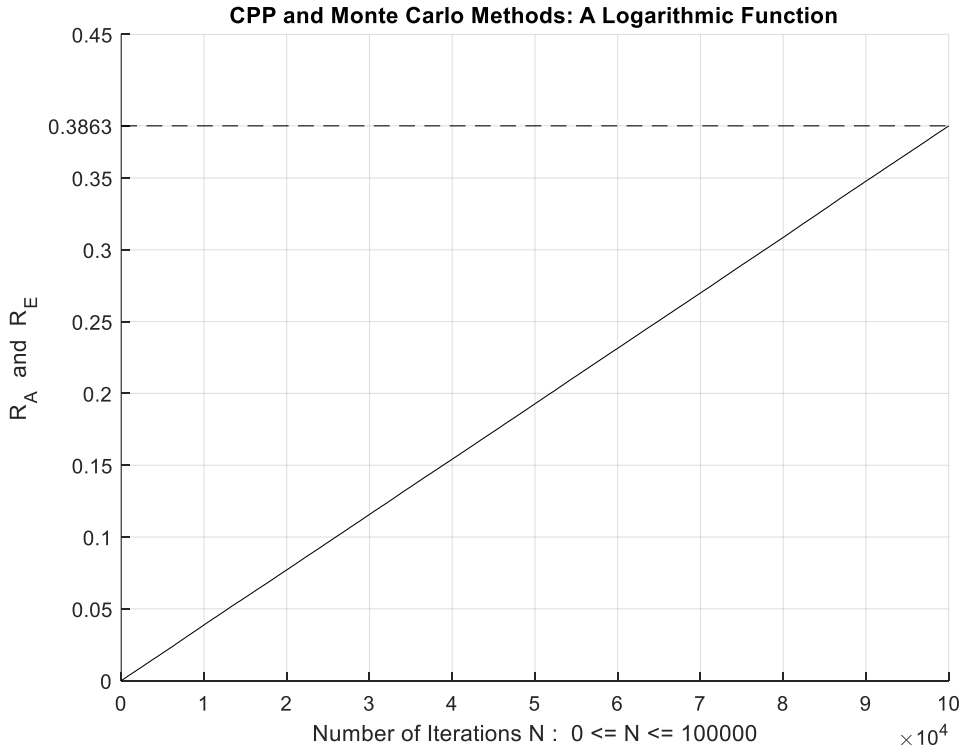


Figure 46. The increasing convergence of the Monte Carlo method up to N = 100,000 iterations.

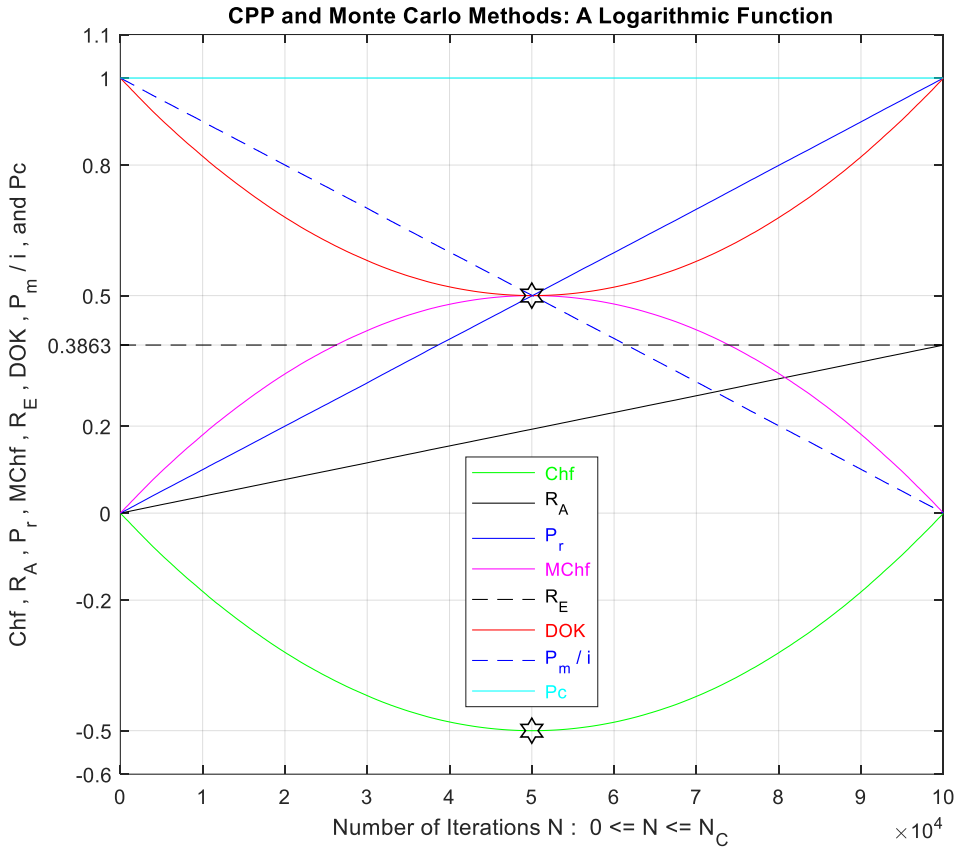


Figure 47. The *CPP* parameters and the Monte Carlo method for a logarithmic function.

IX-1-7- A Multiple Integral

Let us consider the multidimensional integral of the following function:

$$\int_0^{3/2}\int_0^{3/2}\int_0^{3/2} xyz.dxdydz = \int_0^{3/2}\int_0^{3/2}\left[\frac{x^2}{2}\right]_0^{3/2} yz.dydz = \int_0^{3/2}\int_0^{3/2}\frac{9}{8}yz.dydz$$

$$= \frac{9}{8}\int_0^{3/2}\left[\frac{y^2}{2}\right]_0^{3/2} z.dz = \frac{9}{8}\int_0^{3/2}\frac{9}{8}z.dz = \frac{81}{64}\left[\frac{z^2}{2}\right]_0^{3/2}$$

$$= \frac{81}{64}\times\frac{9}{8} = \frac{729}{512} = 1.423828125\ldots$$

$\Leftrightarrow R_E = 1.423828125\ldots$ by the deterministic methods of calculus.

$\Leftrightarrow f(x,y,z) = xyz,\ \ x \mapsto U(0,3/2),\ y \mapsto U(0,3/2),\ z \mapsto U(0,3/2)$

$$\Leftrightarrow \int_0^{3/2}\int_0^{3/2}\int_0^{3/2} xyz.dxdydz \cong \frac{[(3/2-0)\times(3/2-0)\times(3/2-0)]}{N}\sum_{j=1}^{N} x_j y_j z_j$$

$$= \frac{27/8}{N}\sum_{j=1}^{N} x_j y_j z_j = R_A$$

with $1 \le N \le N_C$ after applying Monte Carlo method.

Moreover, the four figures (Figures 48-51) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E = 1.423828125\ldots$ for N = 50, 100, 500, and $N = N_C = 100,000$ iterations. Therefore, we have:

$\lim_{N\to+\infty} P_r(N) = \lim_{N\to+\infty}\left\{1-\left|\frac{R_E - R_A(N)}{R_E}\right|\right\} = 1-\left|\frac{R_E - R_E}{R_E}\right| = 1-0 = 1$ which is equal to the convergence probability of Monte Carlo method as $N \to +\infty$.

Additionally, Figure 52 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters ($Chf, R_A, P_r, MChf, R_E, DOK, P_m / i, Pc$) after applying it to this three-dimensional integral.

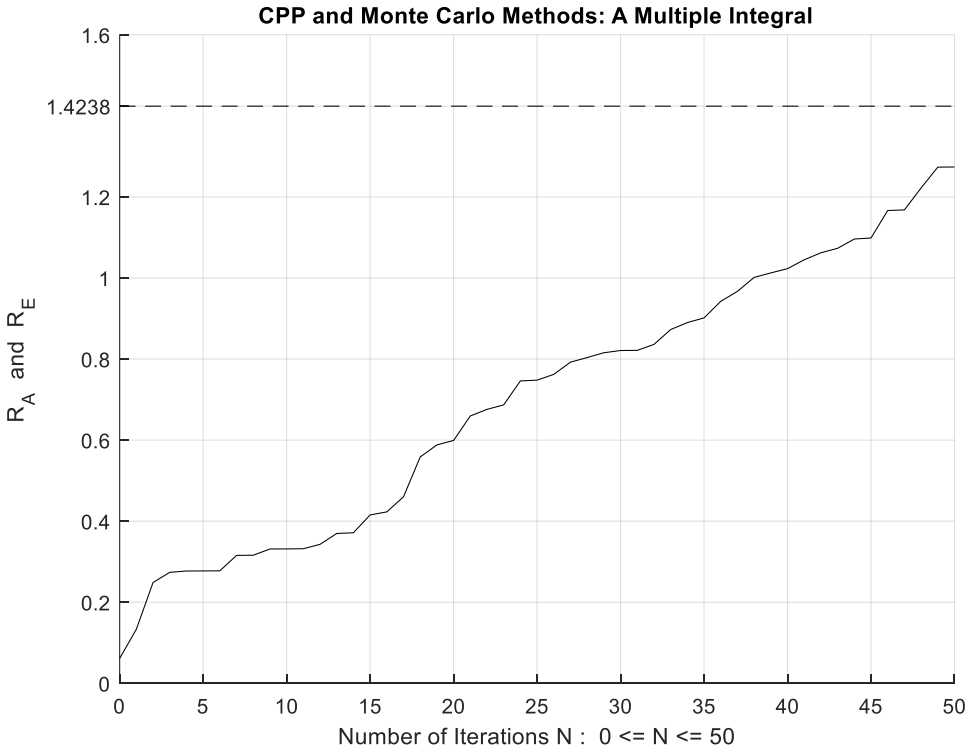


Figure 48. The increasing convergence of the Monte Carlo method up to N = 50 iterations.

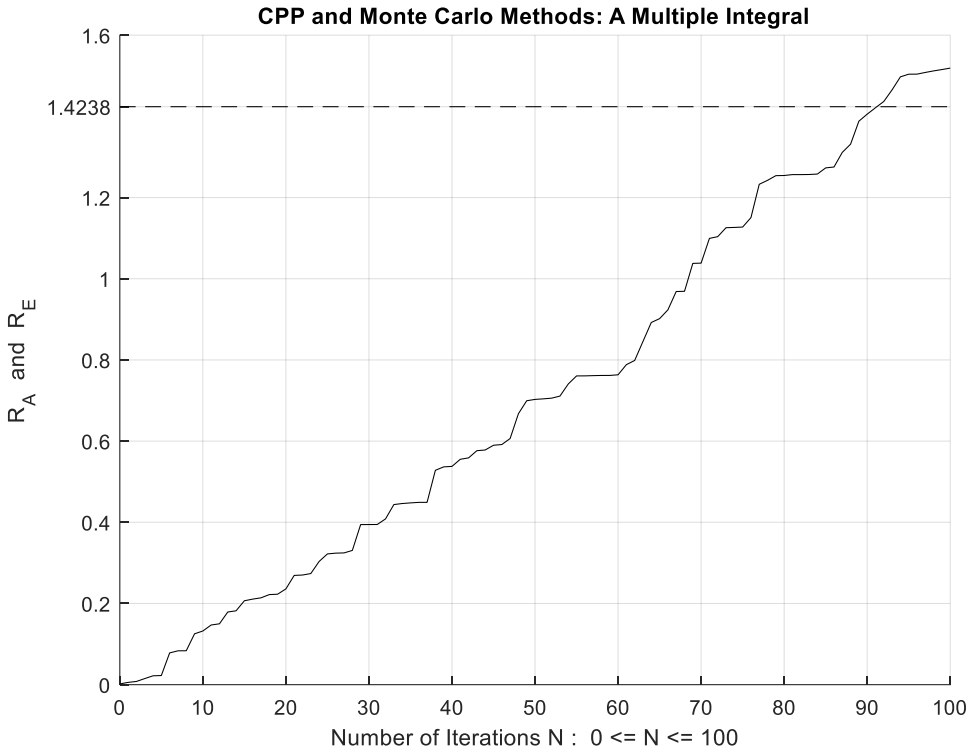


Figure 49. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

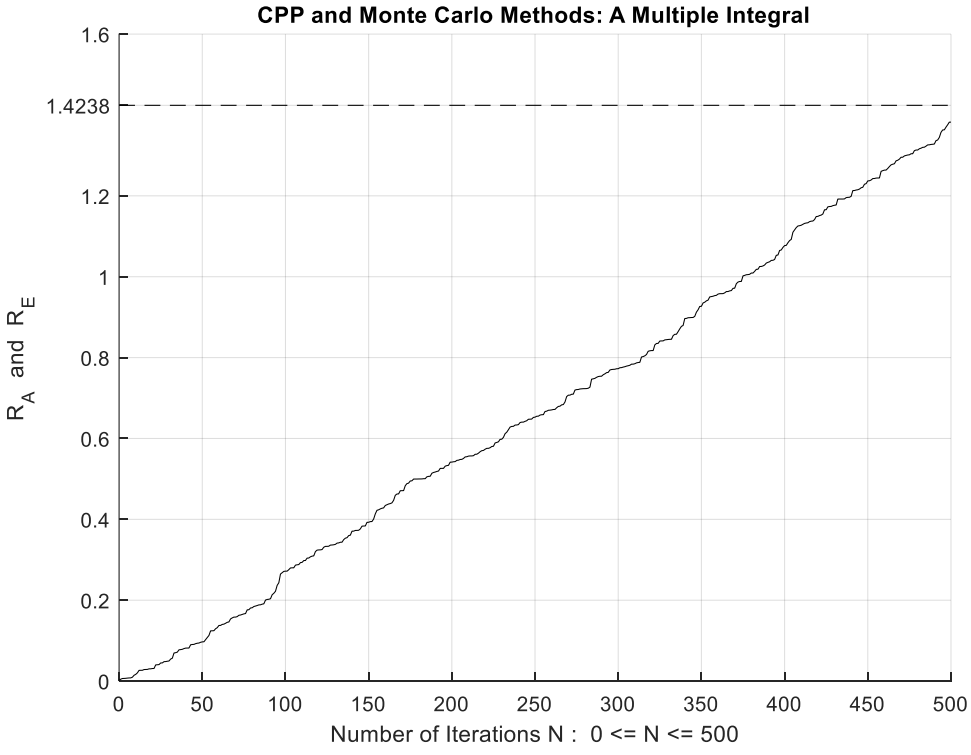


Figure 50. The increasing convergence of the Monte Carlo method up to $N = 500$ iterations.

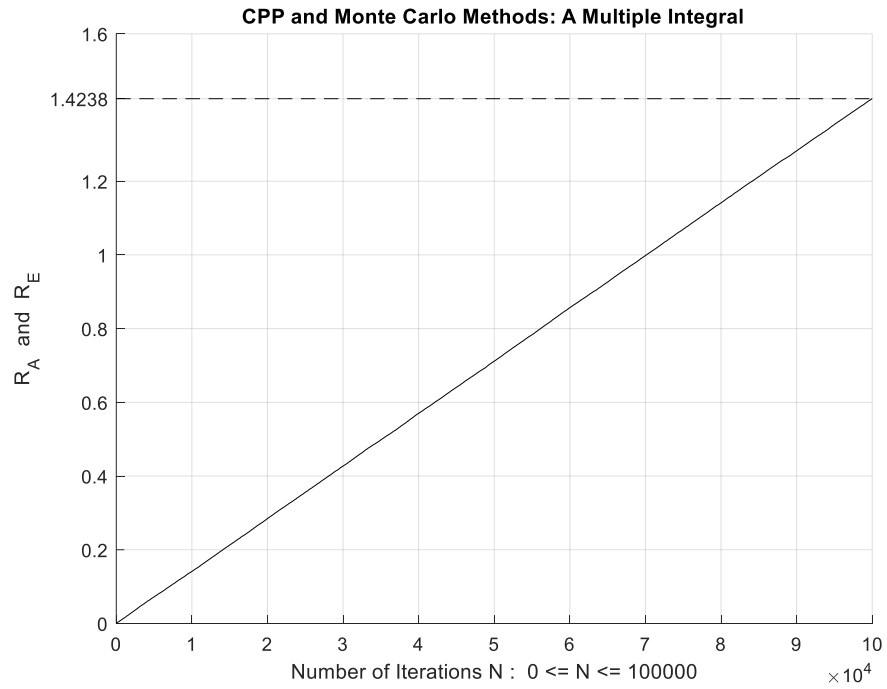


Figure 51. The increasing convergence of the Monte Carlo method up to N = 100,000 iterations.

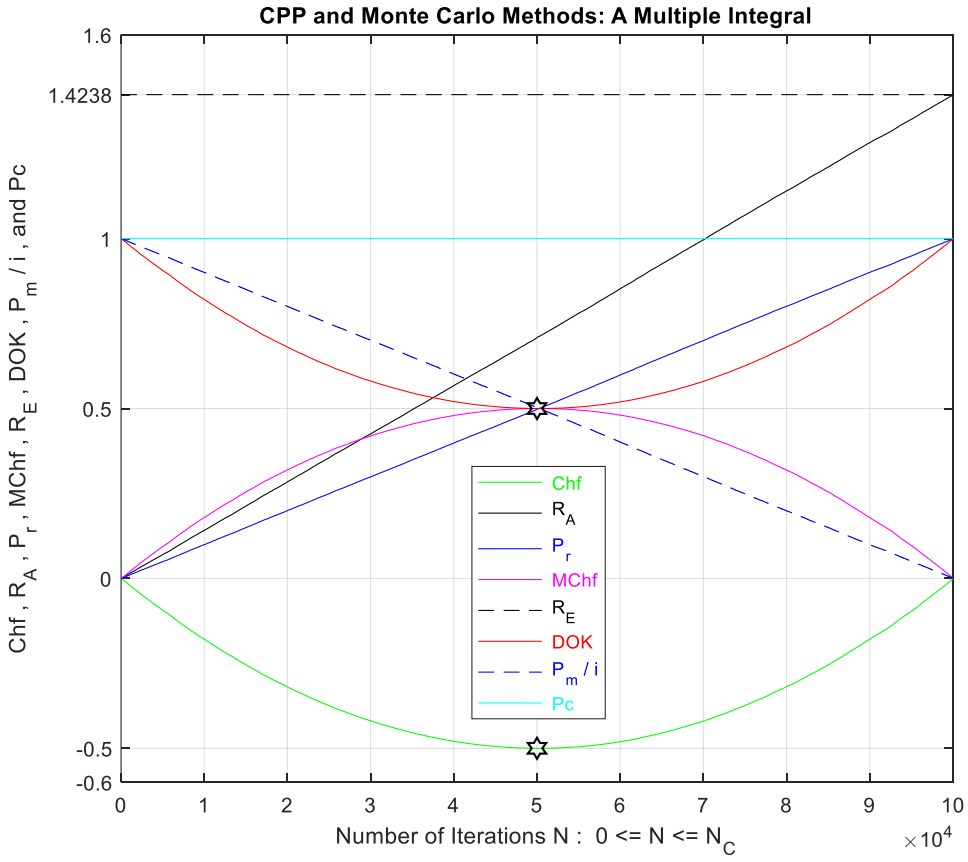


Figure 52. The *CPP* parameters and the Monte Carlo method for a multiple integral.

IX-2- The Discrete Random Case

IX-2-1- The First Random Experiment: A Random Walk in a Plane

We will try in this problem to *simulate random walks in a plane*, each walk starting at **O**(0,0) and consisting of s = 10000 steps of length = L = 0.008. The probability theory says that after s steps, the expected distance from the starting point will be $L\times\sqrt{s}$. So, the estimated distance in the program will be $=0.008\times\sqrt{10000}=0.008\times100=0.8=R_E$. The figure below shows a random walk in a plane (Figure 53):

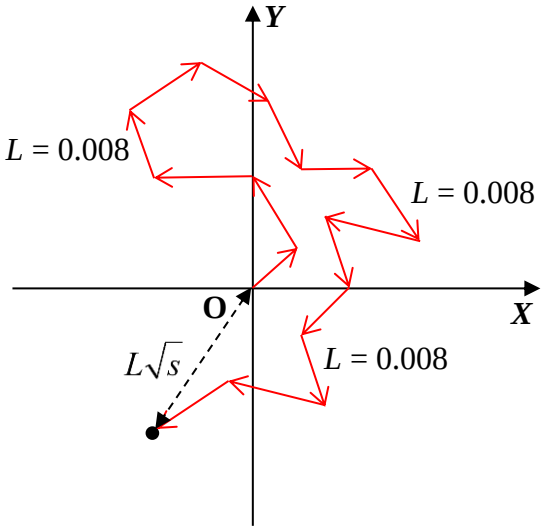


Figure 53. A random walk simulation in a plane

The algorithm in Microsoft Visual C++ is the following:

```
#include <iostream>
#include <cstdlib>
#include <ctime>
#include <cmath>
#include <iomanip>

using namespace std;

int main()
{
	const long double PI = 3.1415926535897931;
	long int i, j, s, k, N;
	long double d, d1, d2, alpha, sum, L, f, F;

	cout << "                          THE RANDOM WALK PROBLEM"
		<< endl;
```

```
        cout <<
             "                          ---------------------------\n"
             << endl;

        srand(time(0));

        N = 100000; L = 0.008;
        for (i = 1; i <= 3; i++)
        {
              sum = 0;
              for (k = 1; k <= N; k++)
              {
                    d = d1 = d2 = 0;
                    s = 10000; alpha = 0; f = 0; F = 0;
                    for (j = 1; j <= s; j++)
                    {
                          f = (long double) rand() / 32767;
                          alpha = 2*PI*f;
                          d1 = d1 + (long double) L*f*cos(alpha);
                          d2 = d2 + (long double) L*f*sin(alpha);
                          F = F + f;
                    }

                    d1 = (long double) d1 / F;
                    d2 = (long double) d2 / F;
                    d = (long double) PI*sqrt(s)*pow((d1*d1)+(d2*d2),0.5);
                    sum = sum + d;
              }

              sum = (long double) sum / N;

              cout << fixed << setprecision(3);
              cout << "AFTER " << s << " STEPS OF LENGTH = " << L
                   << " THE PARTICLE IS AT A DISTANCE " << sum << endl;
              cout << "THE ESTIMATED DISTANCE IS  = " << L*sqrt(s) << "\n"
                   << endl;

        }

        return 0;
}
```

Moreover, the four figures (Figures 54-57) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E = 0.8$ for N = 50, 100, 500, and $N = N_C = 100,000$ iterations. Therefore, we have:

$\lim_{N\to+\infty} P_r(N) = \lim_{N\to+\infty}\left\{1-\left|\frac{R_E - R_A(N)}{R_E}\right|\right\} = 1-\left|\frac{R_E - R_E}{R_E}\right| = 1-0 = 1$ which is equal to the convergence probability of Monte Carlo method as $N \to +\infty$.

Additionally, Figure 58 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters ($Chf, R_A, P_r, MChf, R_E, DOK, P_m / i, Pc$) after applying it to this random walk problem.

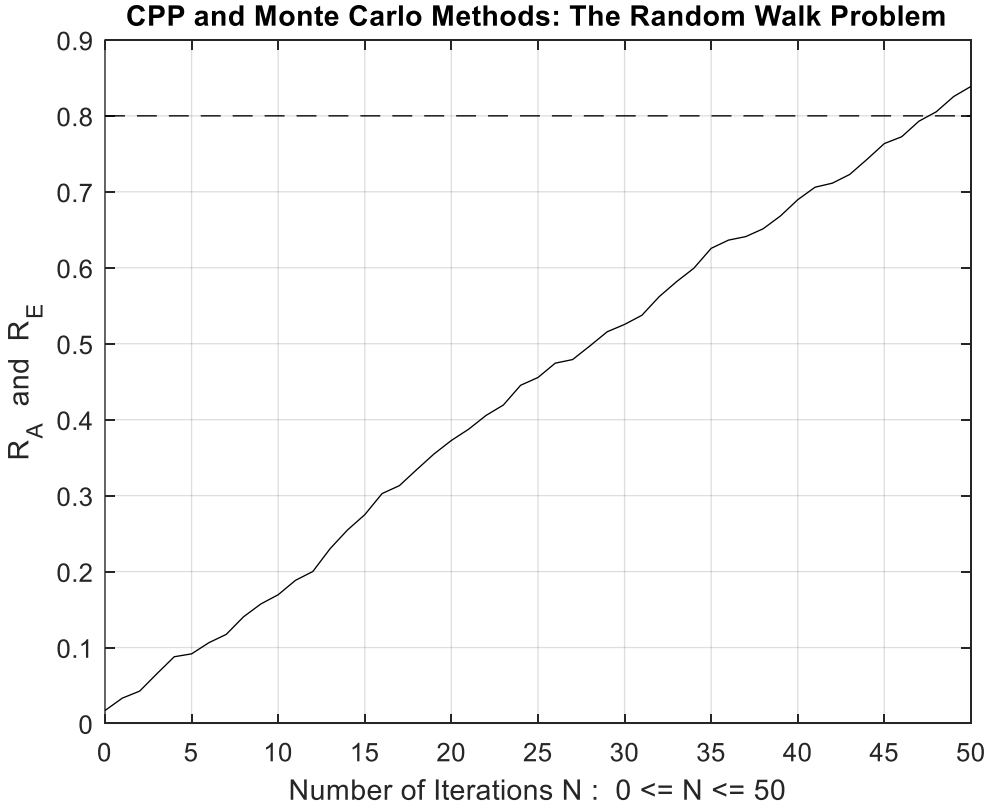


Figure 54. The increasing convergence of the Monte Carlo method up to N = 50 iterations.

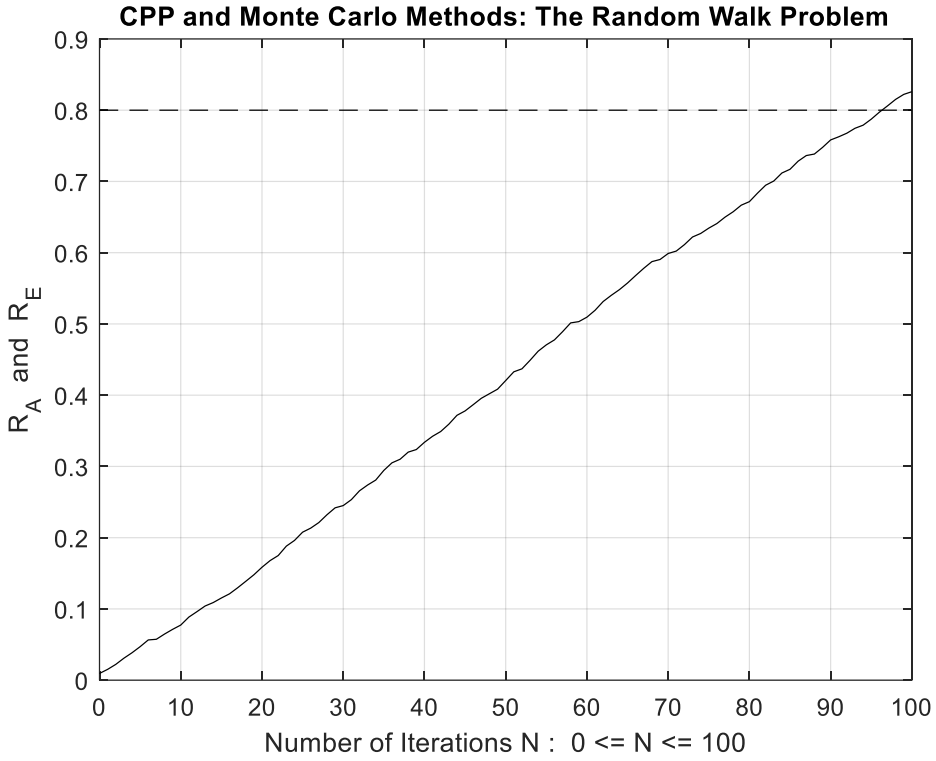


Figure 55. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

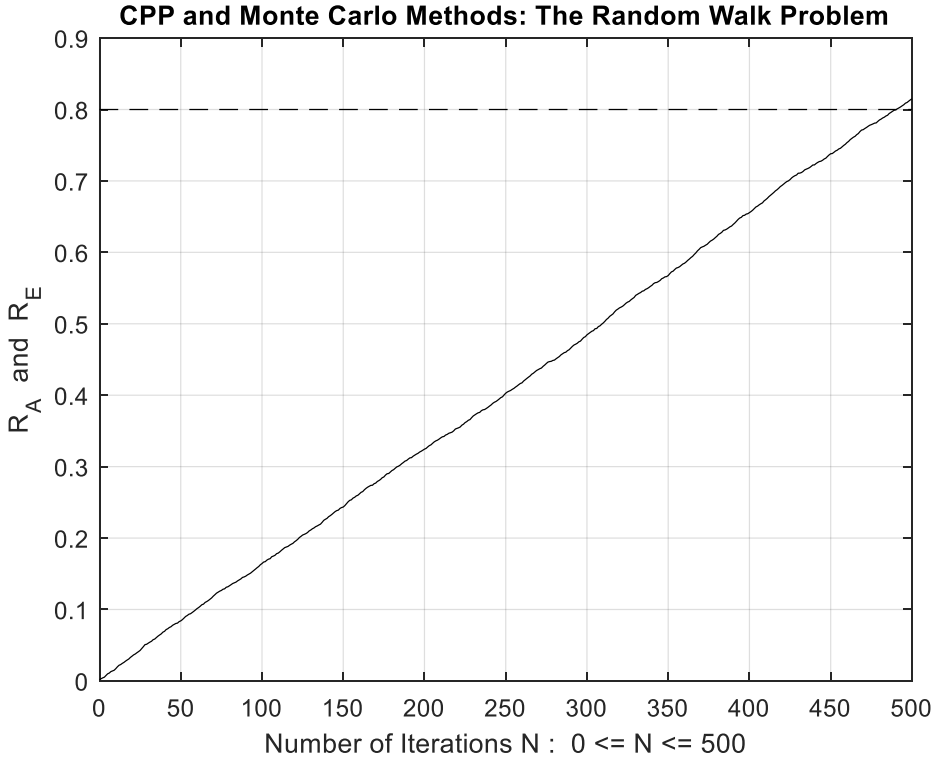


Figure 56. The increasing convergence of the Monte Carlo method up to $N = 500$ iterations.

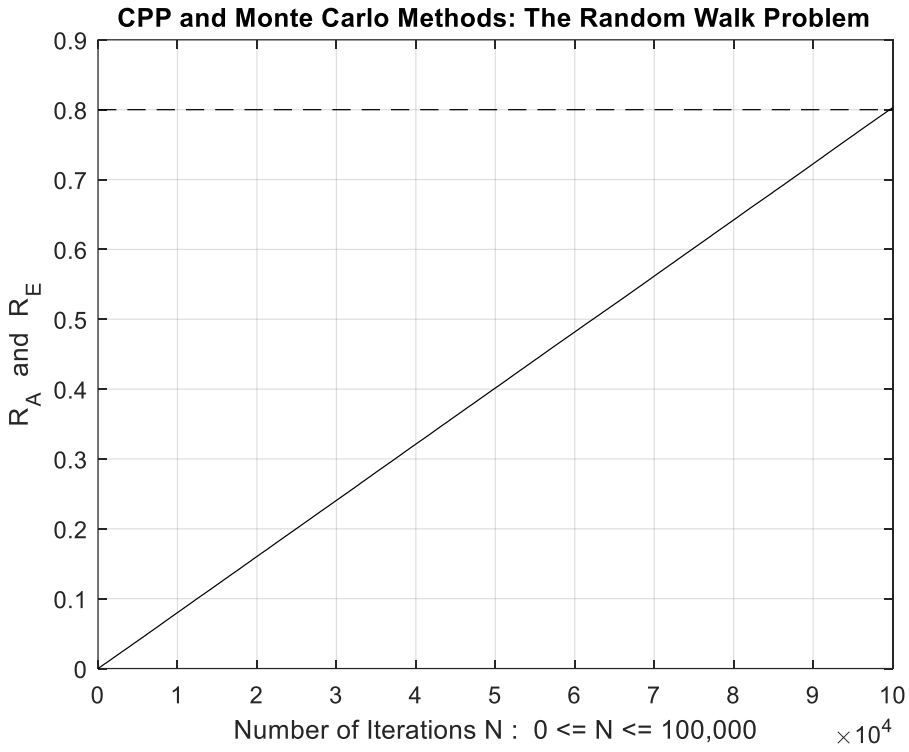


Figure 57. The increasing convergence of the Monte Carlo method up to N = 100,000 iterations.

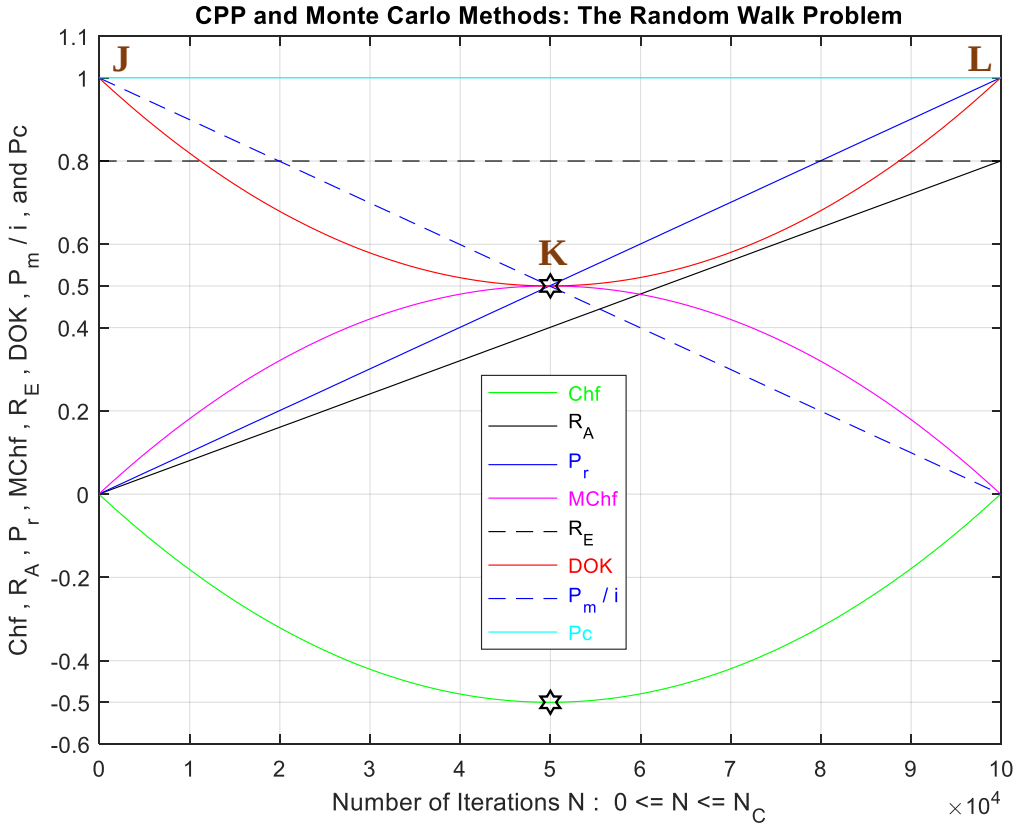


Figure 58. The *CPP* parameters and the Monte Carlo method for the random walk problem.

IX-2-1-1 The Complex Probability Cubes

In the first cube (Figure 59), the simulation of *DOK* and *Chf* as functions of each other and of the iterations *N* for the random walk problem can be seen. The line in cyan is the projection of $Pc^2(N) = DOK(N) - Chf(N) = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point J (*DOK* = 1, *Chf* = 0) when $N = 0$ iterations, reaches the point (*DOK* = 0.5, *Chf* = -0.5) when $N =$ 50,000 iterations, and returns at the end to J (*DOK* = 1, *Chf* = 0) when $N = N_C = 100{,}000$ iterations. The other curves are the graphs of *DOK*(*N*) (red) and *Chf*(*N*) (green, blue, pink) in different planes. Notice that they all have a minimum at the point K (*DOK* = 0.5, *Chf* = -0.5, *N* = 50,0000 iterations). The point L corresponds to (*DOK* = 1, *Chf* = 0, $N = N_C = 100{,}000$ iterations). The three points J, K, L are the same as in Figure 58.

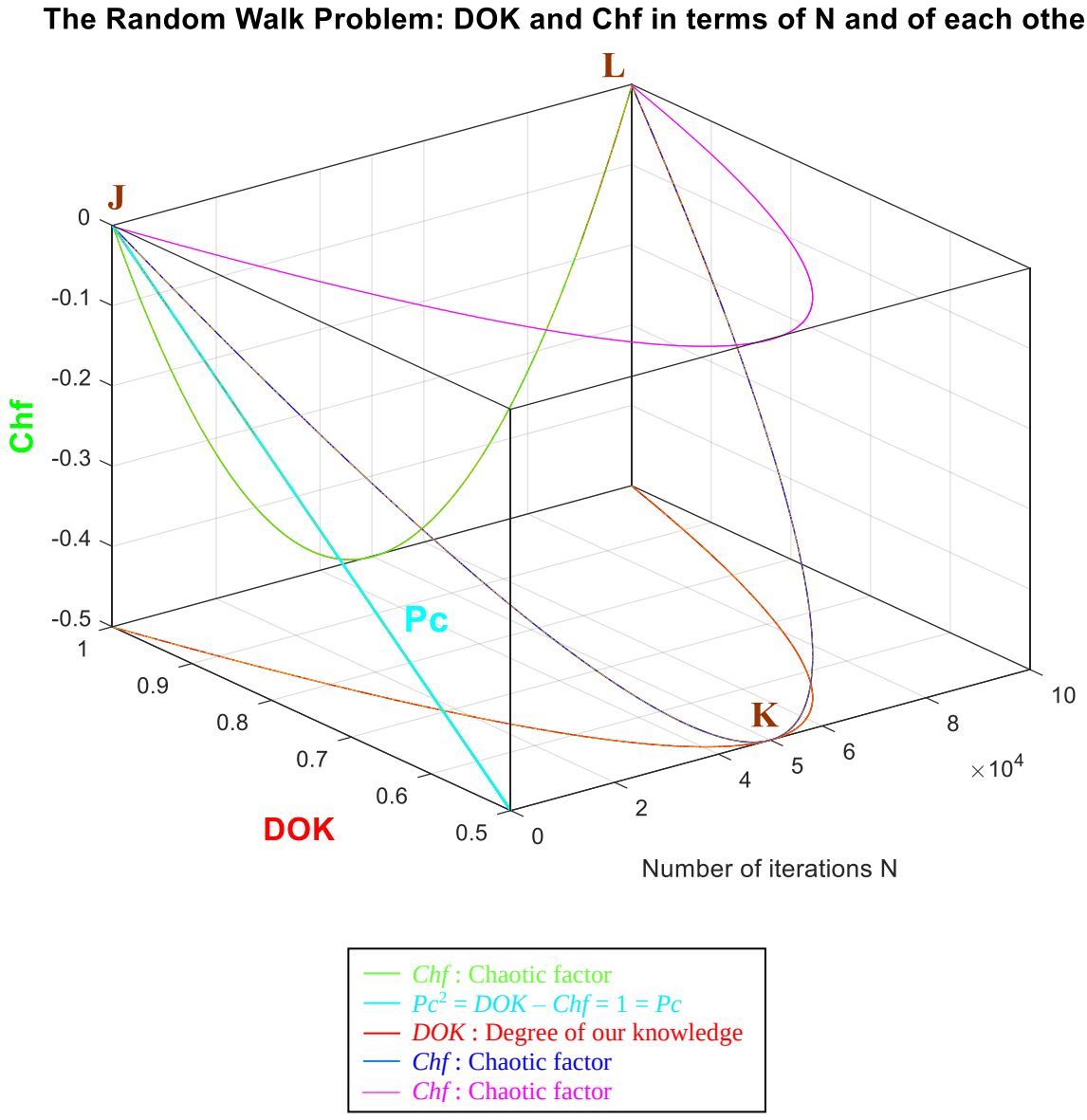


Figure 59. *DOK* and *Chf* in terms of *N* and of each other for the random walk problem.

In the second cube (Figure 60), we can notice the simulation of the convergence probability $P_r(N)$ and its complementary real divergence probability $P_m(N)/i$ in terms of the iterations N for the random walk problem. The line in cyan is the projection of $Pc^2(N) = P_r(N) + P_m(N)/i = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point ($P_r = 0$, $P_m/i = 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $P_r(N)$ in the plane $P_r(N) = P_m(N)/i$. This curve starts at the point J ($P_r = 0$, $P_m/i = 1$, $N = 0$ iterations), reaches the point K ($P_r = 0.5$, $P_m/i = 0.5$, $N = 50{,}000$ iterations), and gets at the end to L ($P_r = 1$, $P_m/i = 0$, $N = N_C = 100{,}000$ iterations). The blue curve represents $P_m(N)/i$ in the plane $P_r(N) + P_m(N)/i = 1$. Notice the importance of the point K which is the intersection of the red and blue curves at $N = 50{,}000$ iterations and when $P_r(N) = P_m(N)/i = 0.5$. The three points J, K, L are the same as in Figure 58.

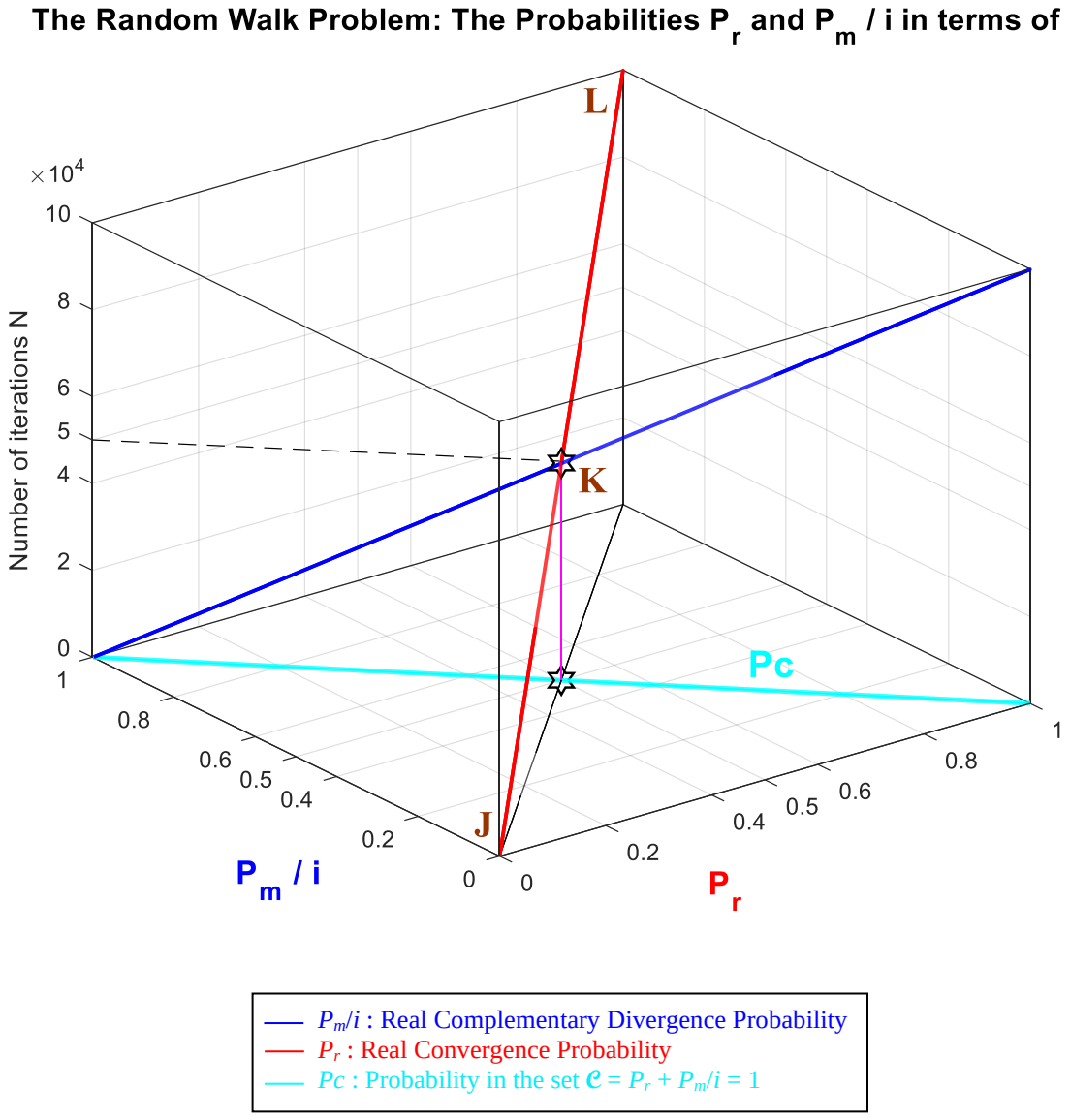


Figure 60- P_r and P_m/i in terms of N and of each other for the random walk problem.

In the third cube (Figure 61), we can notice the simulation of the complex random vector $Z(N)$ in $\mathcal{C}$ as a function of the real convergence probability $P_r(N)$ = Re(Z) in $\mathcal{R}$ and of its complementary imaginary divergence probability $P_m(N) = i\times\text{Im}(Z)$ in $\mathcal{M}$, and this in terms of the iterations N for the random walk problem. The red curve represents $P_r(N)$ in the plane $P_m(N) = 0$ and the blue curve represents $P_m(N)$ in the plane $P_r(N) = 0$. The green curve represents the complex probability vector $Z(N) = P_r(N) + P_m(N) = \text{Re}(Z) + i\times\text{Im}(Z)$ in the plane $P_r(N) = iP_m(N) + 1$. The curve of $Z(N)$ starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point L ($P_r = 1$, $P_m = 0$, $N = N_C = 100{,}000$ iterations). The line in cyan is $P_r(0) = iP_m(0) + 1$ and it is the projection of the $Z(N)$ curve on the complex probability plane whose equation is $N = 0$ iterations. This projected line starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point ($P_r = 1$, $P_m = 0$, $N = 0$ iterations). Notice the importance of the point K corresponding to $N = 50{,}000$ iterations and when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in Figure 58.

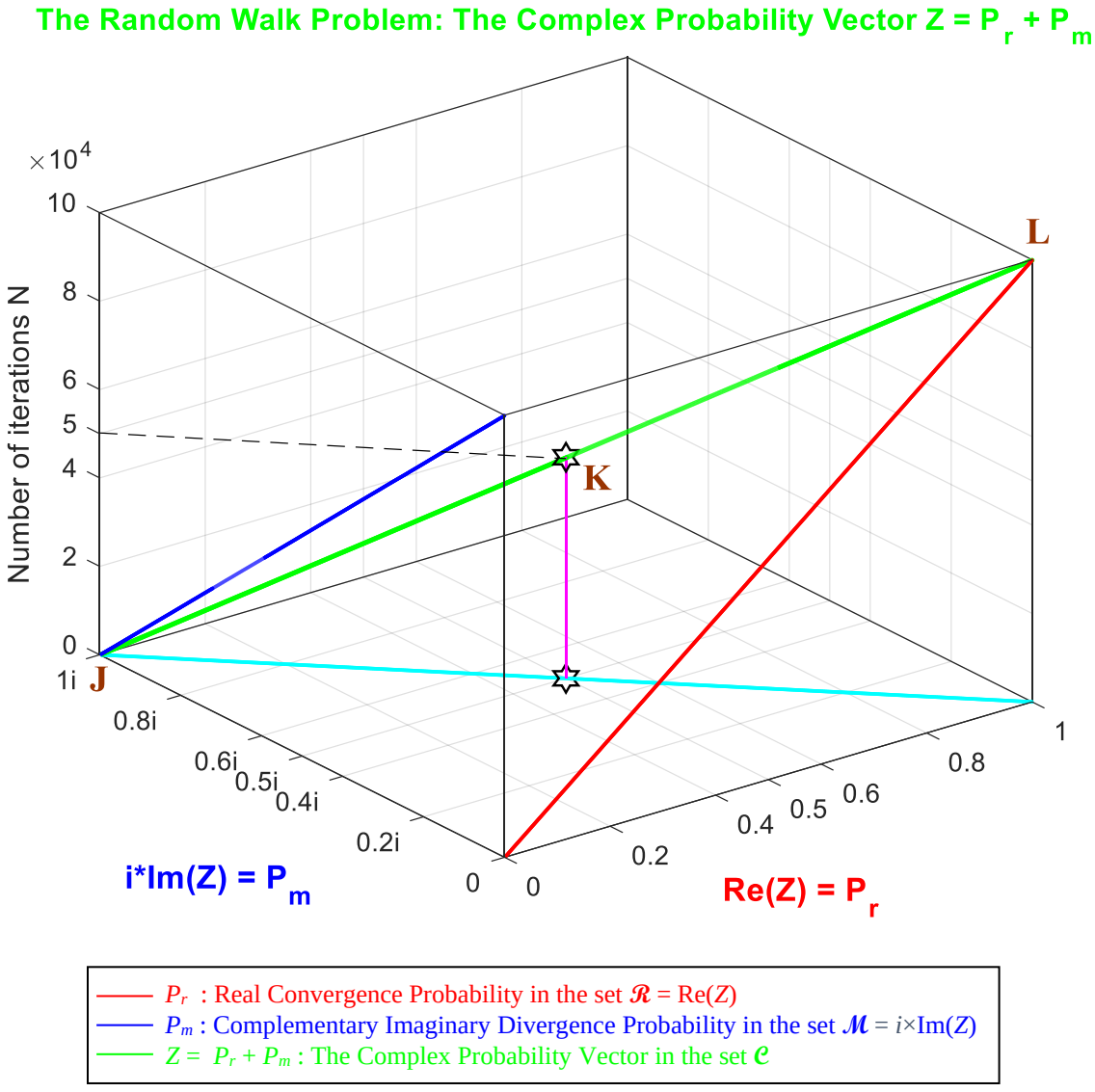


Figure 61- The Complex Probability Vector Z in terms of N for the random walk problem.

IX-2-2- The Second Random Experiment: The Birthday Problem

The given of the second random experiment is the following: Find the probability that n people $(n \le 365)$ selected at random will have n different birthdays.

Theoretical Analysis

We assume that there are only 365 days in a year (not a leap year) and that all birthdays are equally probable, assumptions which are not quite met in reality.

The first of the n people has of course some birthday with probability 365/365 = 1. Then, if the second is to have a different birthday, it must occur on one of the other days. Therefore, the probability that the second person has a birthday different from the first is 364/365. Similarly, the probability that the third person has a birthday different from the first two is 363/365. Finally, the probability that the nth person has a birthday different from the others is $(365-n+1)/365$. We therefore have:

$$P(\text{all } n \text{ birthdays are different}) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \ldots \times \left(1 - \frac{n-1}{365}\right) = R_E$$

The table below gives the theoretical probabilities of different birthdays for a selected number of people n (Table 3).

Number of People n	Theoretical Probability = R_E
$n = 1$	$P = 1$
$n = 2$	$P = 0.99726$
$n = 3$	$P = 0.991796$
$n = 4$	$P = 0.983644$
$n = 5$	$P = 0.972864$
$n = 10$	$P = 0.883052$
$n = 20$	$P = 0.588562$
$n = 50$	$P = 0.0296264$
$n = 75$	$P = 0.000280122$
$n = 100$	$P = 0.000000307249$
$n > 365$	$P = 0$

Table 3. The theoretical probabilities of distinct birthdays for n people where $n \ge 1$.

The algorithm in Microsoft Visual C++ is the following:

```
#include <iostream>
#include <iomanip>
#include <cstdlib>
#include <ctime>
#include <cmath>
```

```
using namespace std;

void theoretical(int);
void simulated(int);

int main()
{
      int n;
      cout << "                                THE BIRTHDAY PROGRAM"
           << endl;
      cout << "                               ----------------------\n"
           << endl;

      n = 13;

      cout << fixed << setprecision(4);

      theoretical(n);

      cout << "WAIT.....................................\n"
           << endl;

      simulated(n);

      return 0;
}

void theoretical(int n)
{
      int i;
      long double prod;
      long double P;

      prod = 1;
      for (i = 0; i <= (n - 1); i++)
            prod = (long double) prod * (365 - i) / 365;

      P = prod;

      cout << "The theoretical probability of " << n
           << " distinct birthday(s) is = " << P << "\n" << endl;
}

void simulated(int n)
{
```

```
        int table[366] = { 0 };
        long int random, match;
        long double P, sum, N;
        int i, j;

        srand(time(0));

        N = 500000000;

        sum = 0; random = 0; P = 0; match = 0;
        for (j = 1; j <= N; j++)
        {
              for (i = 1; i <= n; i++)
              {
                    random = 1 + rand() % 365;
                    table[random] += 1;
                    random = 0;
              }

              i = 0;
              while ((i <= 365) && (match == 0))
              {
                    if (table[i] >= 2)
                          match = 1;
                    ++i;
              }

              if (match == 0)
                    sum += 1;

              for (i = 0; i <= 365; i++)
                    table[i] = 0;

              match = 0;
        }

        P = (long double) sum / N;

        cout << "The simulated   probability of " << n
             << " distinct birthday(s) is = " << P << "\n" << endl;
}
```

Moreover, the four figures (Figures 62-65) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E = 0.80558972\ldots$ for $n = 13$ people and for N = 50, 100, 500, and $N = N_C = 500,000,000$ iterations. Therefore, we have:

$\lim_{N\to+\infty} P_r(N) = \lim_{N\to+\infty}\left\{1-\left|\frac{R_E - R_A(N)}{R_E}\right|\right\} = 1-\left|\frac{R_E - R_E}{R_E}\right| = 1-0 = 1$ which is equal to the convergence probability of Monte Carlo method as $N \to +\infty$.

Additionally, Figure 66 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters ($Chf, R_A, P_r, MChf, R_E, DOK, P_m / i, Pc$) after applying it to this birthday problem.

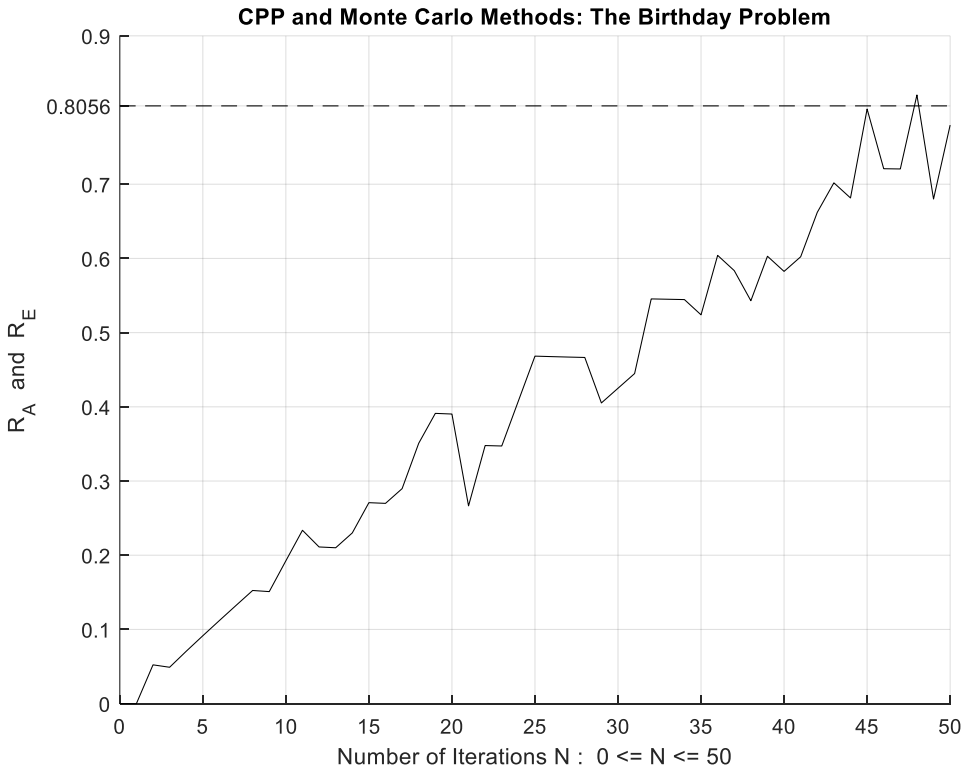


Figure 62. The increasing convergence of the Monte Carlo method up to N = 50 iterations.

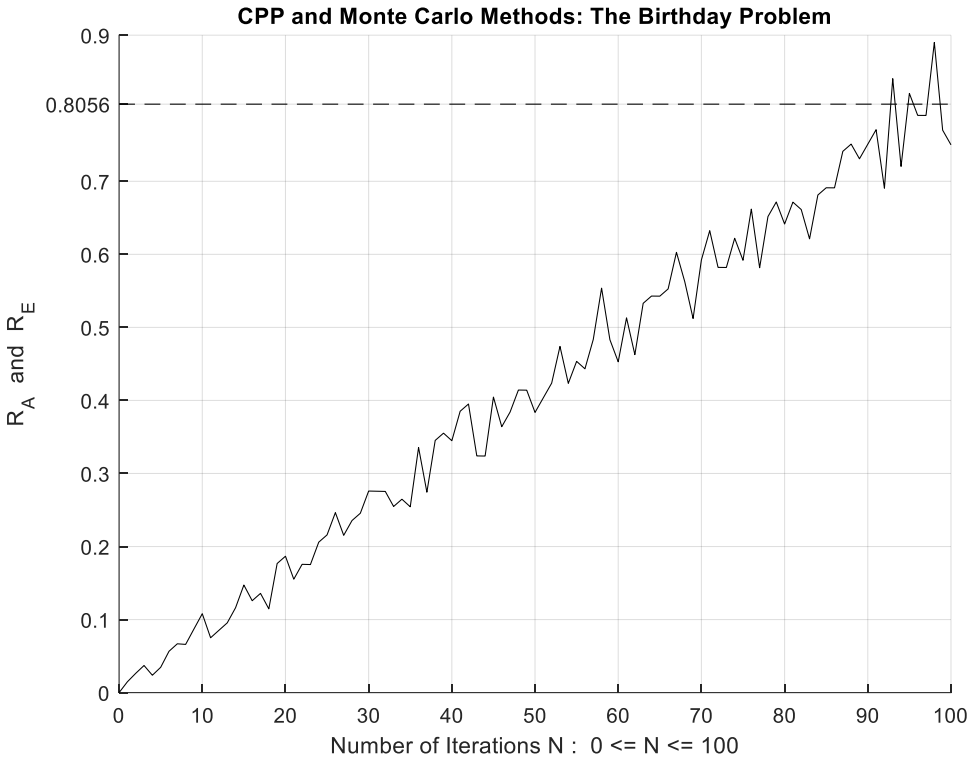


Figure 63. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

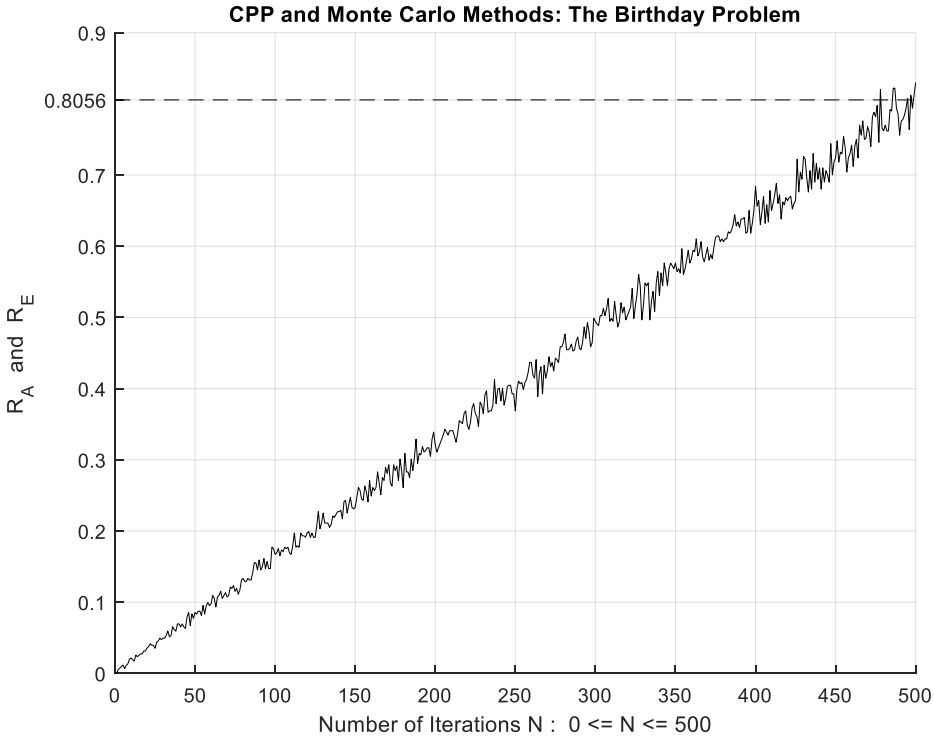


Figure 64. The increasing convergence of the Monte Carlo method up to $N = 500$ iterations.

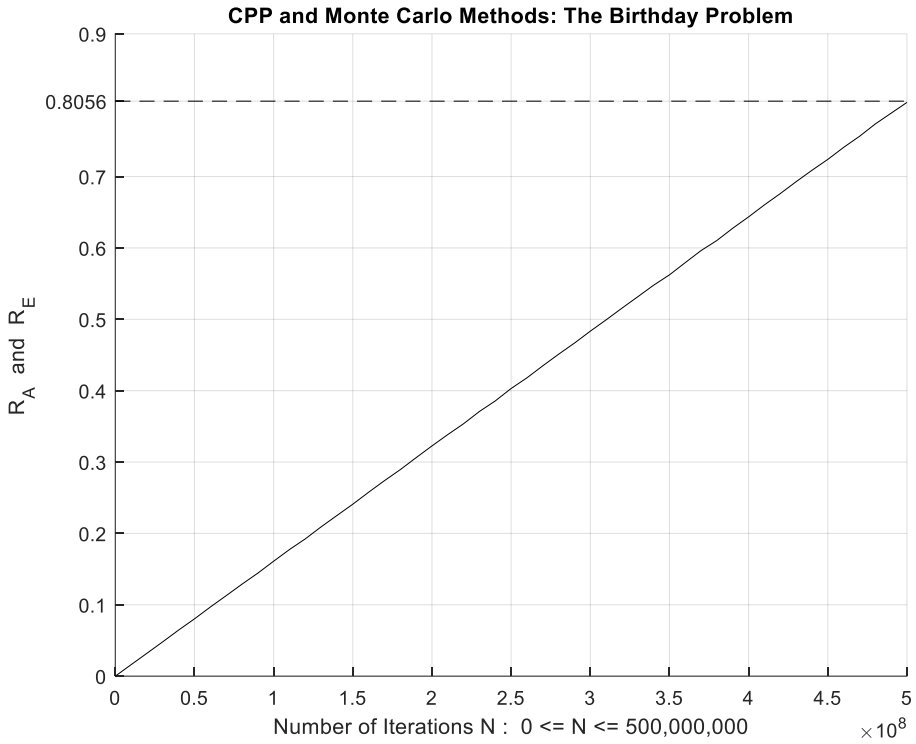


Figure 65. The increasing convergence of the Monte Carlo method up to N = 500,000,000 iterations.

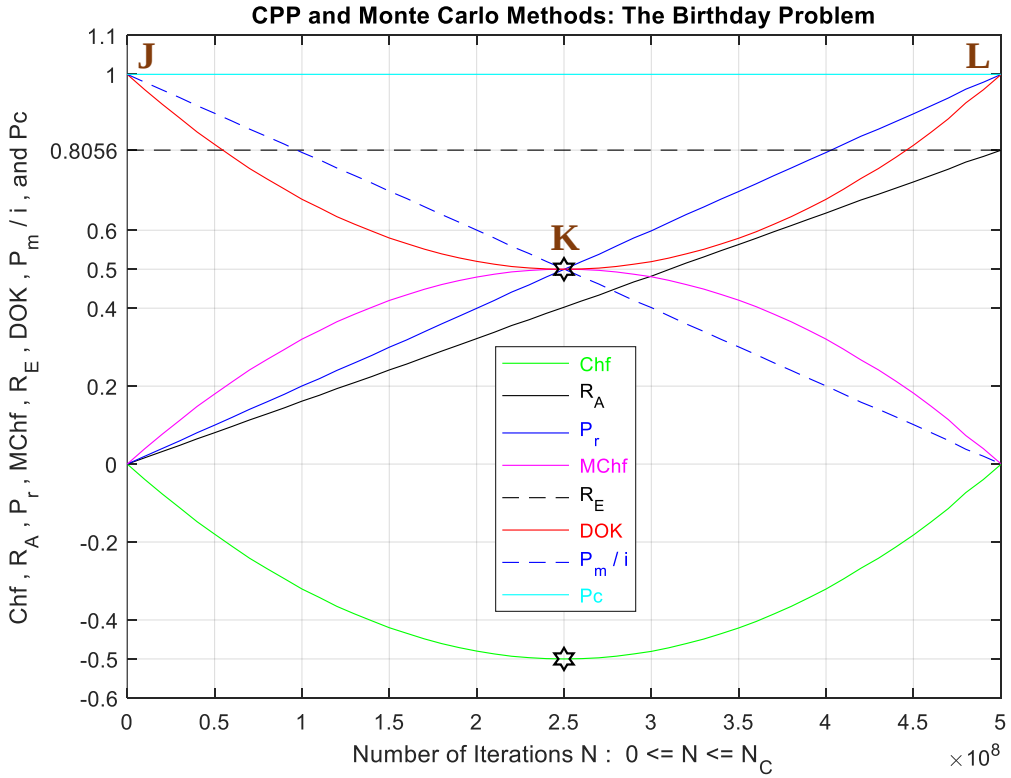


Figure 66. The *CPP* parameters and the Monte Carlo method for the birthday problem.

IX-2-2-1 The Complex Probability Cubes

In the first cube (Figure 67), the simulation of *DOK* and *Chf* as functions of each other and of the iterations *N* for the birthday problem can be seen. The line in cyan is the projection of $Pc^2(N) = DOK(N) - Chf(N) = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point J ($DOK = 1$, $Chf = 0$) when $N = 0$ iterations, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when $N = 250{,}000{,}000$ iterations, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $N = N_C = 500{,}000{,}000$ iterations. The other curves are the graphs of $DOK(N)$ (red) and $Chf(N)$ (green, blue, pink) in different planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, $N = 250{,}000{,}000$ iterations). The point L corresponds to ($DOK = 1$, $Chf = 0$, $N = N_C = 500{,}000{,}000$ iterations). The three points J, K, L are the same as in Figure 66.

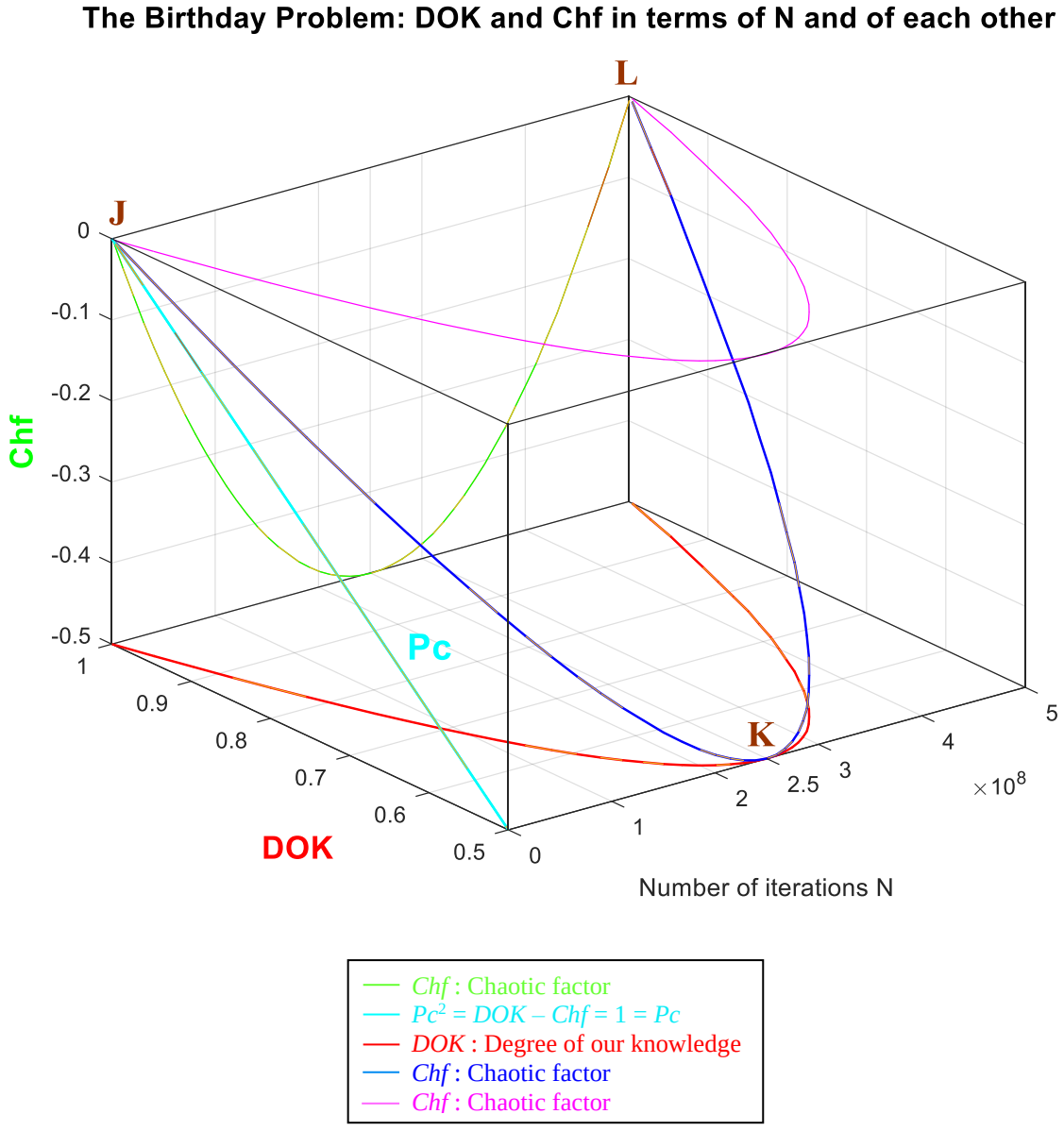


Figure 67. *DOK* and *Chf* in terms of *N* and of each other for the birthday problem.

In the second cube (Figure 68), we can notice the simulation of the convergence probability $P_r(N)$ and its complementary real divergence probability $P_m(N)/i$ in terms of the iterations N for the birthday problem. The line in cyan is the projection of $Pc^2(N) = P_r(N) + P_m(N)/i = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point ($P_r = 0$, $P_m/i = 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $P_r(N)$ in the plane $P_r(N) = P_m(N)/i$. This curve starts at the point J ($P_r = 0$, $P_m/i = 1$, $N = 0$ iterations), reaches the point K ($P_r = 0.5$, $P_m/i = 0.5$, $N =$ 250,000,000 iterations), and gets at the end to L ($P_r = 1$, $P_m/i = 0$, $N = N_C =$ 500,000,000 iterations). The blue curve represents $P_m(N)/i$ in the plane $P_r(N) + P_m(N)/i = 1$. Notice the importance of the point K which is the intersection of the red and blue curves at $N =$ 250,000,000 iterations and when $P_r(N) = P_m(N)/i = 0.5$. The three points J, K, L are the same as in Figure 66.

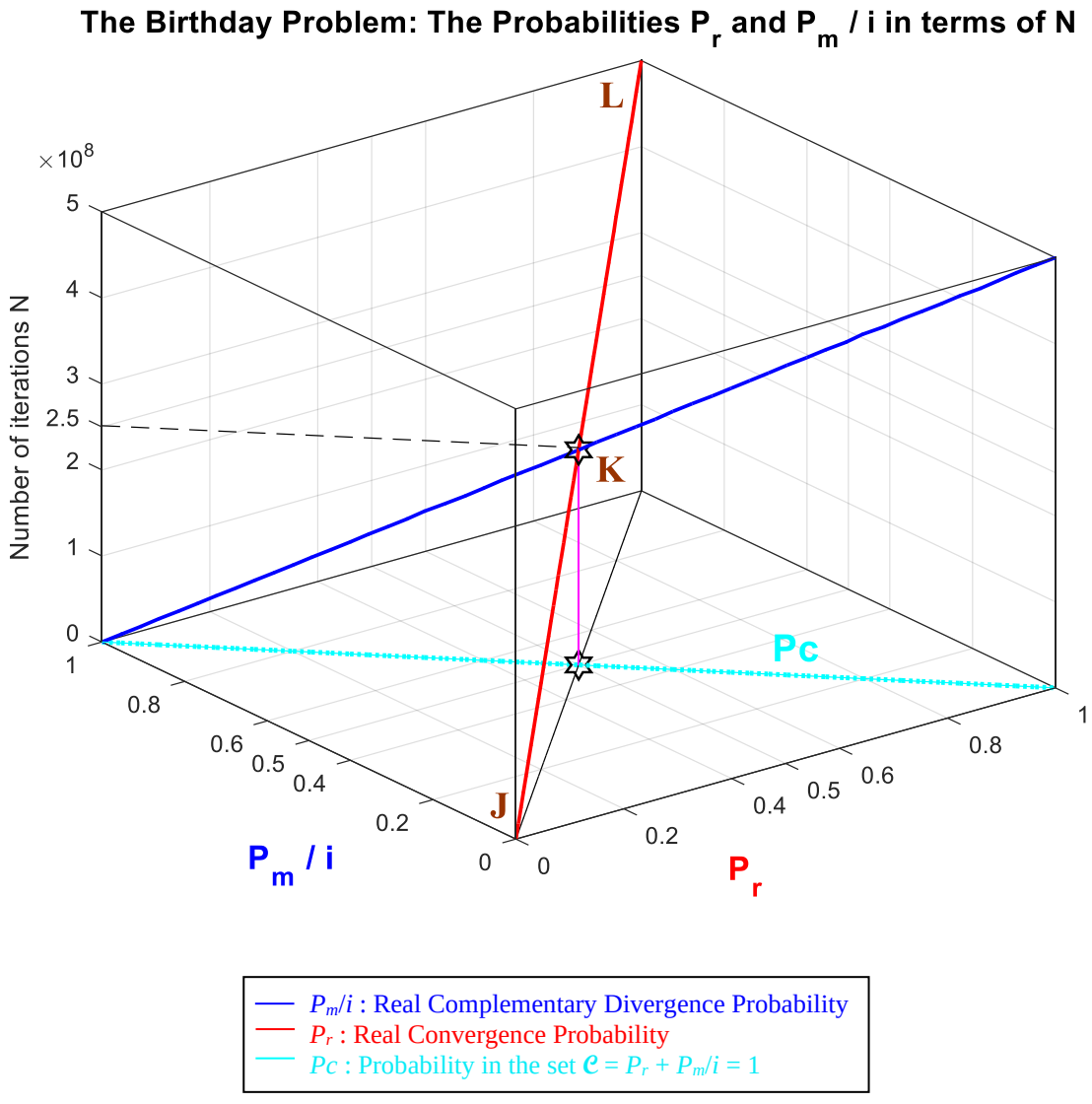


Figure 68. P_r and P_m/i in terms of N and of each other for the birthday problem.

In the third cube (Figure 69), we can notice the simulation of the complex random vector $Z(N)$ in $\mathcal{C}$ as a function of the real convergence probability $P_r(N) = \mathrm{Re}(Z)$ in $\mathcal{R}$ and of its complementary imaginary divergence probability $P_m(N) = i\times\mathrm{Im}(Z)$ in $\mathcal{M}$, and this in terms of the iterations N for the birthday problem. The red curve represents $P_r(N)$ in the plane $P_m(N) = 0$ and the blue curve represents $P_m(N)$ in the plane $P_r(N) = 0$. The green curve represents the complex probability vector $Z(N) = P_r(N) + P_m(N) = \mathrm{Re}(Z) + i\times\mathrm{Im}(Z)$ in the plane $P_r(N) = iP_m(N) + 1$. The curve of $Z(N)$ starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point L ($P_r = 1$, $P_m = 0$, $N = N_C = 500{,}000{,}000$ iterations). The line in cyan is $P_r(0) = iP_m(0) + 1$ and it is the projection of the $Z(N)$ curve on the complex probability plane whose equation is $N = 0$ iterations. This projected line starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point ($P_r = 1$, $P_m = 0$, $N = 0$ iterations). Notice the importance of the point K corresponding to $N = 250{,}000{,}000$ iterations and when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in Figure 66.

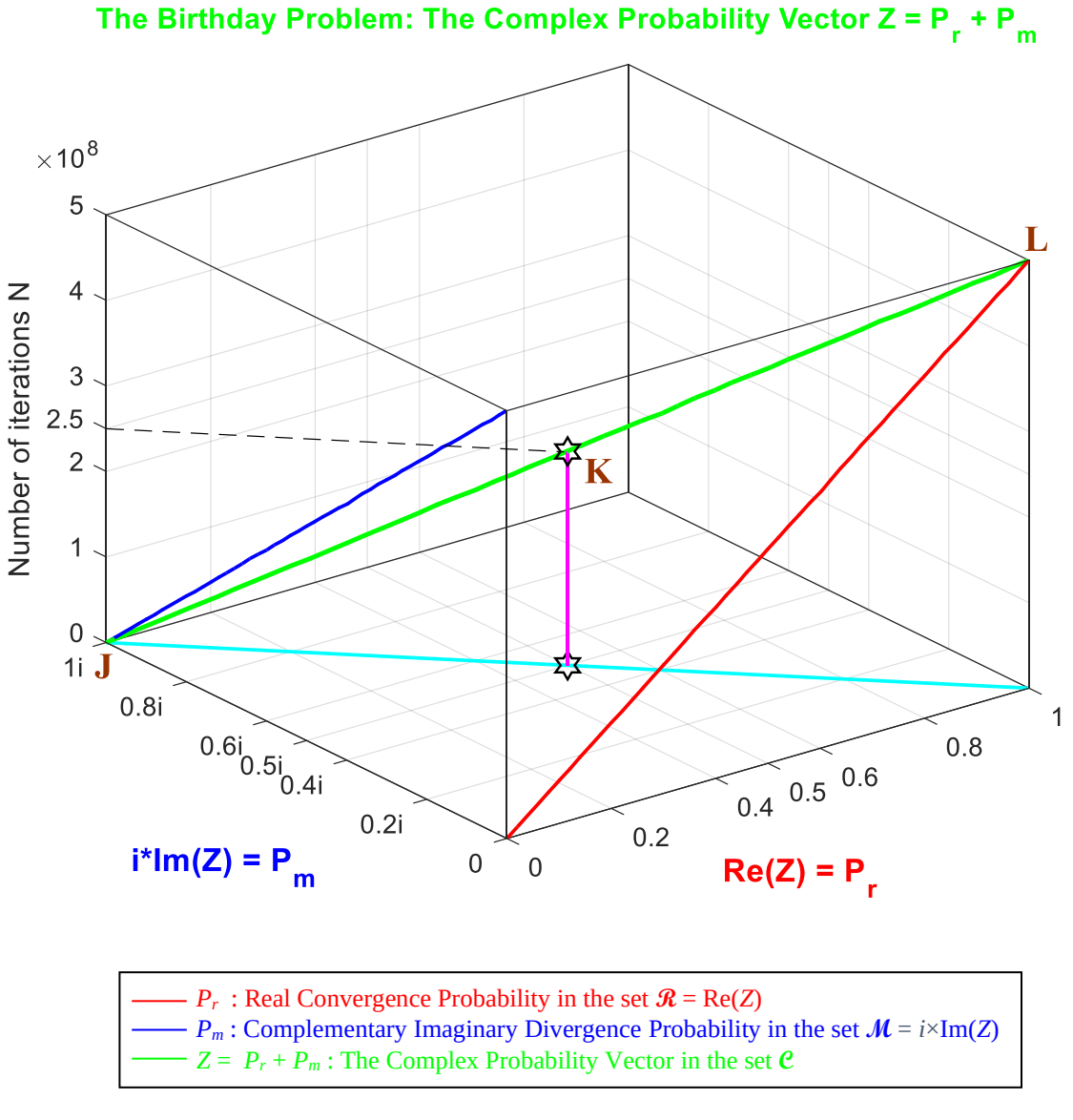


Figure 69. The Complex Probability Vector Z in terms of N for the birthday problem.

IX-2-3- The Third Random Experiment: The Two Dice Problem

The following program has an analytic solution beside a simulated solution. This is advantageous for us because we wish to compare the results of Monte Carlo simulations with theoretical solutions. Consider the experiment of tossing two dice. For an unloaded die, the numbers 1,2,3,4,5, and 6 are equally likely to occur. We ask: *What is the probability of throwing a* 12 (*i.e.*, 6 *appearing on each die*) *in* 14 *throws of the dice*?

There are six possible outcomes from each die for a total of 36 possible combinations. Only one of these combinations is a double 6, so 35 out of the 36 combinations are not correct. With 14 throws, we have $\left(\frac{35}{36}\right)^{14}$ as the probability of a wrong outcome. Hence, $1-\left(\frac{35}{36}\right)^{14} = 0.325910425\ldots$ is the exact answer and therefore the value of R_E. Not all random problems of this type can be analyzed like this.

The algorithm in Microsoft Visual C++ is the following:

```
#include <iostream>
#include <iomanip>
#include <cstdlib>
#include <ctime>
#include <cmath>

using namespace std;

void theoretical(void);
void simulated(void);

int main()
{
      cout << "                              THE TWO DICE PROBLEM"
           << endl;
      cout << "                            ------------------------\n"
           << endl;

      cout << fixed << setprecision(4);

      theoretical();
      simulated();

      cout << endl;

      return 0;
}
```

```
void theoretical(void)
{
      long double P, p;

      p = (long double) 35 / 36;
      P = 1 - (long double) pow(p, 14);

      cout <<
      "The theoretical probability of throwing a (6,6) in 14 throws = "
           << P << "\n" << endl;
}
void simulated(void)
{
      long int i, j, die1, die2, sum;
      long int counter, N;
      long double P;

      srand(time(0));

      N = 100000000;

      sum = 0; counter = 0;
      for (i = 1; i <= N; i++)
      {
            j = 1; sum = 0;
            while ((j <= 14) && (sum != 12))
            {
                  die1 = 1 + rand() % 6;
                  die2 = 1 + rand() % 6;
                  sum = die1 + die2;

                  if (sum == 12)
                        ++counter;
                  ++j;
            }
      }

      P = (long double) counter / N;
      cout <<
   "\nThe simulated   probability of throwing a (6,6) in 14 throws = "
           << P << "\n" << endl;
}
```

Moreover, the four figures (Figures 70-73) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E = 0.325910425\ldots$ for N = 50, 100, 500, and $N = N_C = 100,000,000$ iterations. Therefore, we have:

$\lim_{N\to+\infty} P_r(N) = \lim_{N\to+\infty}\left\{1-\left|\frac{R_E - R_A(N)}{R_E}\right|\right\} = 1-\left|\frac{R_E - R_E}{R_E}\right| = 1-0 = 1$ which is equal to the convergence probability of Monte Carlo method as $N \to +\infty$.

Additionally, Figure 74 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters $(Chf, R_A, P_r, MChf, R_E, DOK, P_m / i, Pc)$ after applying it to this two dice problem.

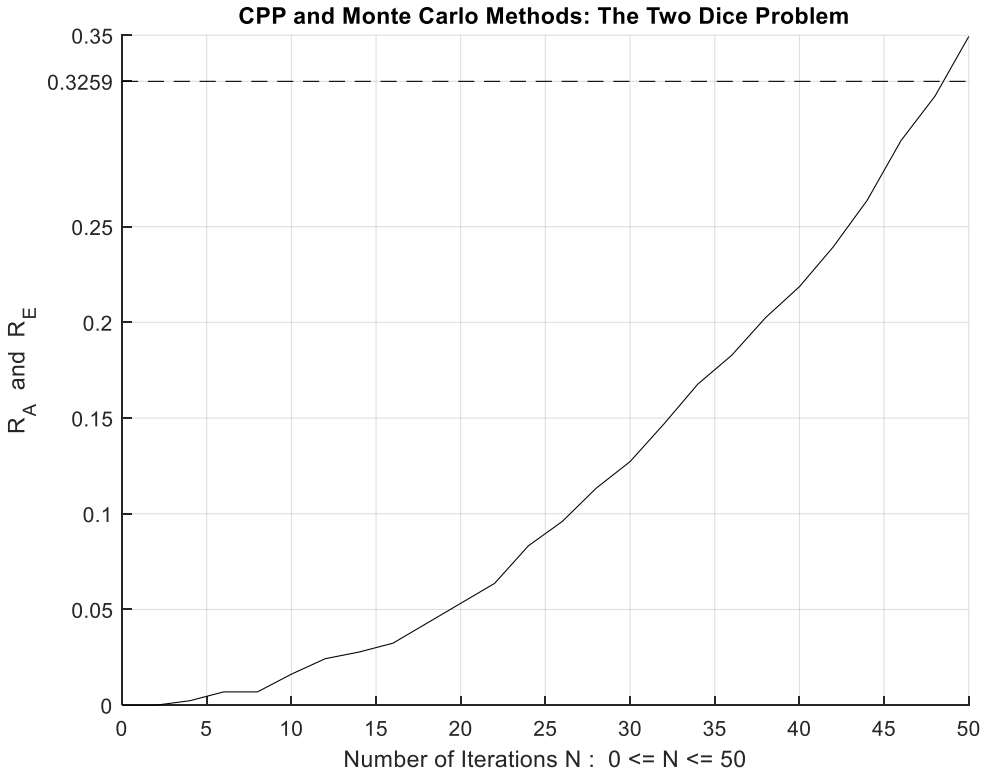


Figure 70. The increasing convergence of the Monte Carlo method up to N = 50 iterations.

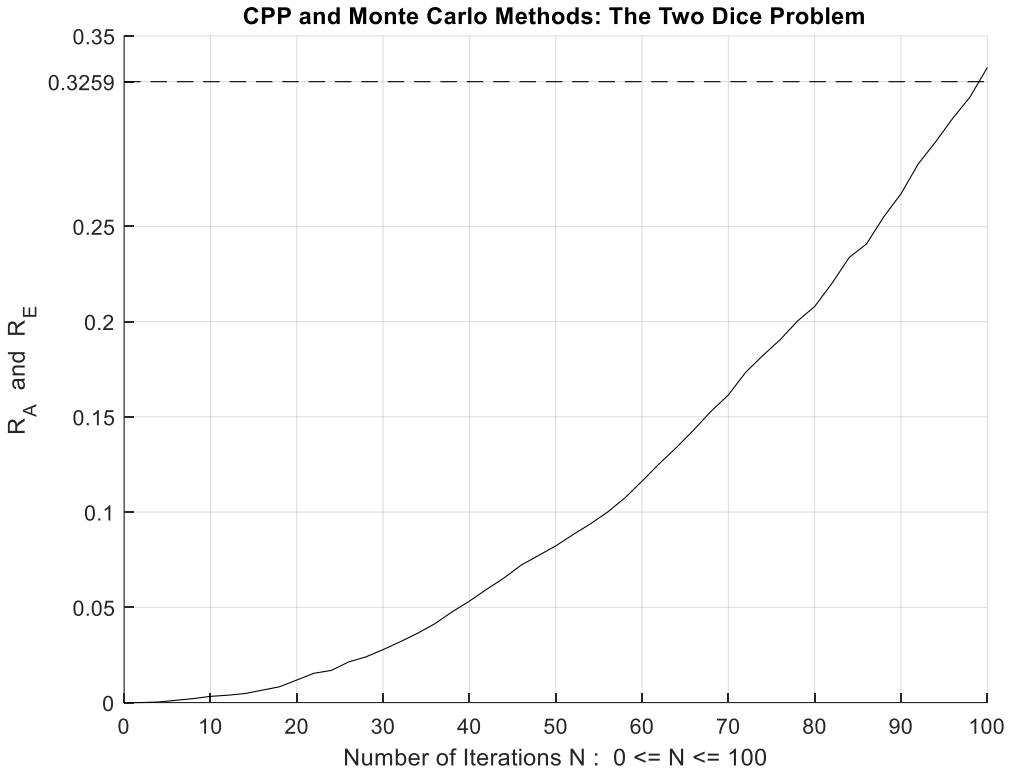


Figure 71. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

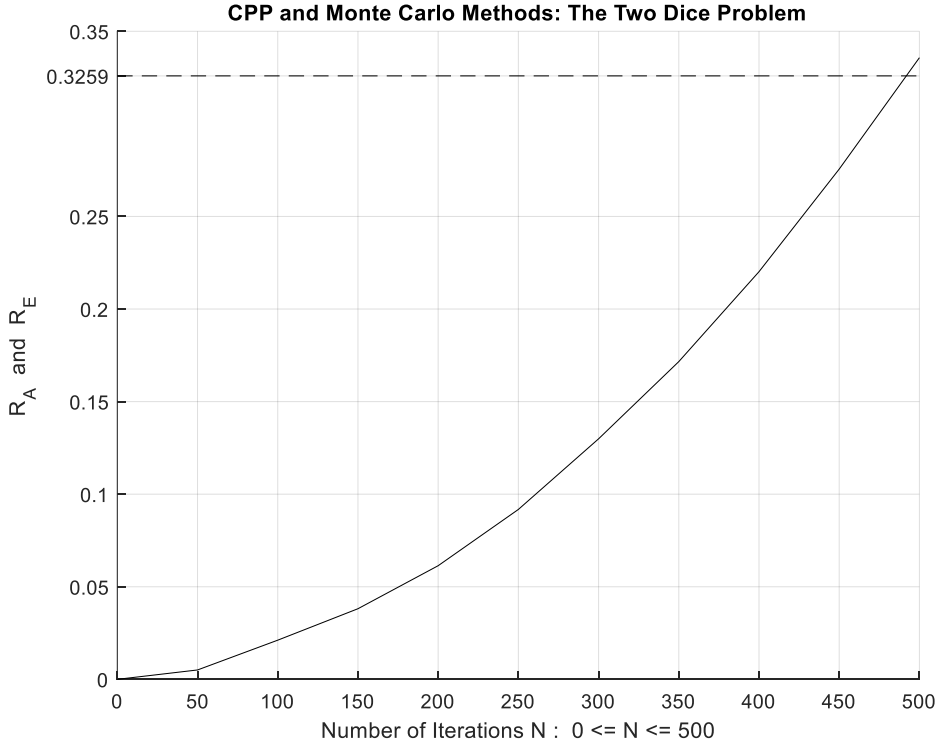


Figure 72. The increasing convergence of the Monte Carlo method up to $N = 500$ iterations.

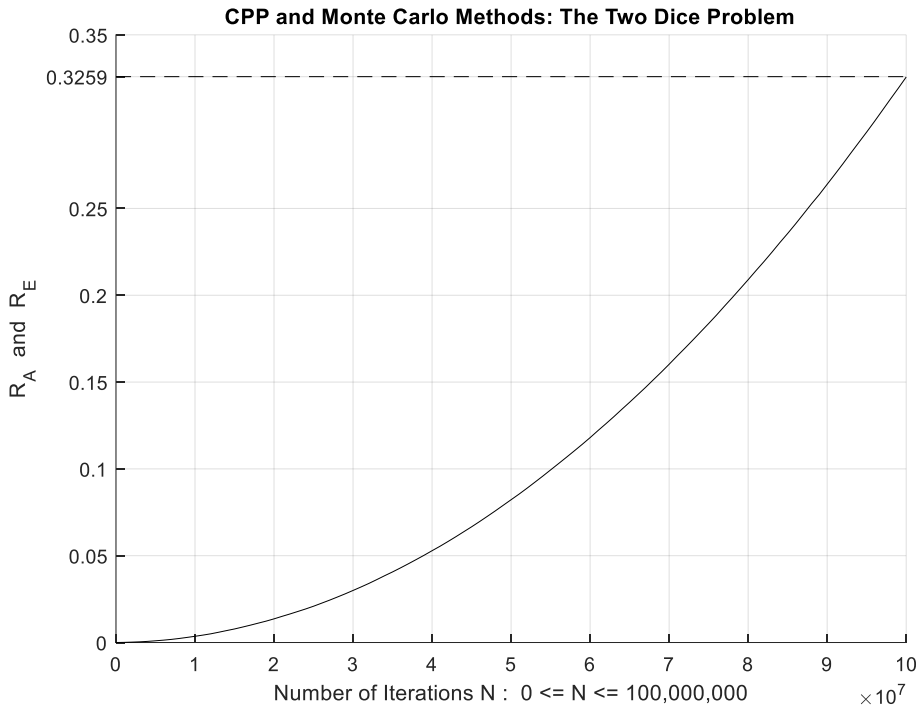


Figure 73. The increasing convergence of the Monte Carlo method up to N = 100,000,000 iterations.

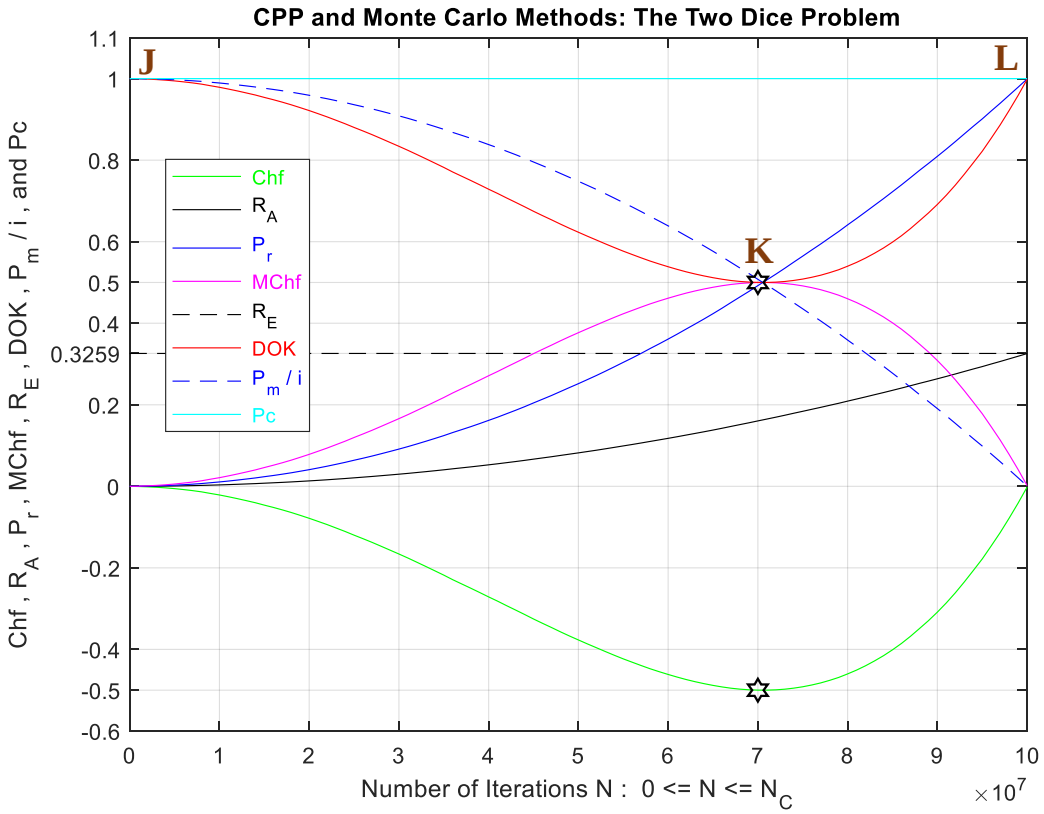


Figure 74. The *CPP* parameters and the Monte Carlo method for the two dice problem.

IX-2-3-1 The Complex Probability Cubes

In the first cube (Figure 75), the simulation of *DOK* and *Chf* as functions of each other and of the iterations *N* for the two dice problem can be seen. The line in cyan is the projection of $Pc^2(N) = DOK(N) - Chf(N) = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point J ($DOK = 1$, $Chf = 0$) when $N = 0$ iterations, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when $N = 70{,}000{,}000$ iterations, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $N = N_C = 100{,}000{,}000$ iterations. The other curves are the graphs of $DOK(N)$ (red) and $Chf(N)$ (green, blue, pink) in different planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, $N = 70{,}000{,}000$ iterations). The point L corresponds to ($DOK = 1$, $Chf = 0$, $N = N_C = 100{,}000{,}000$ iterations). The three points J, K, L are the same as in Figure 74.

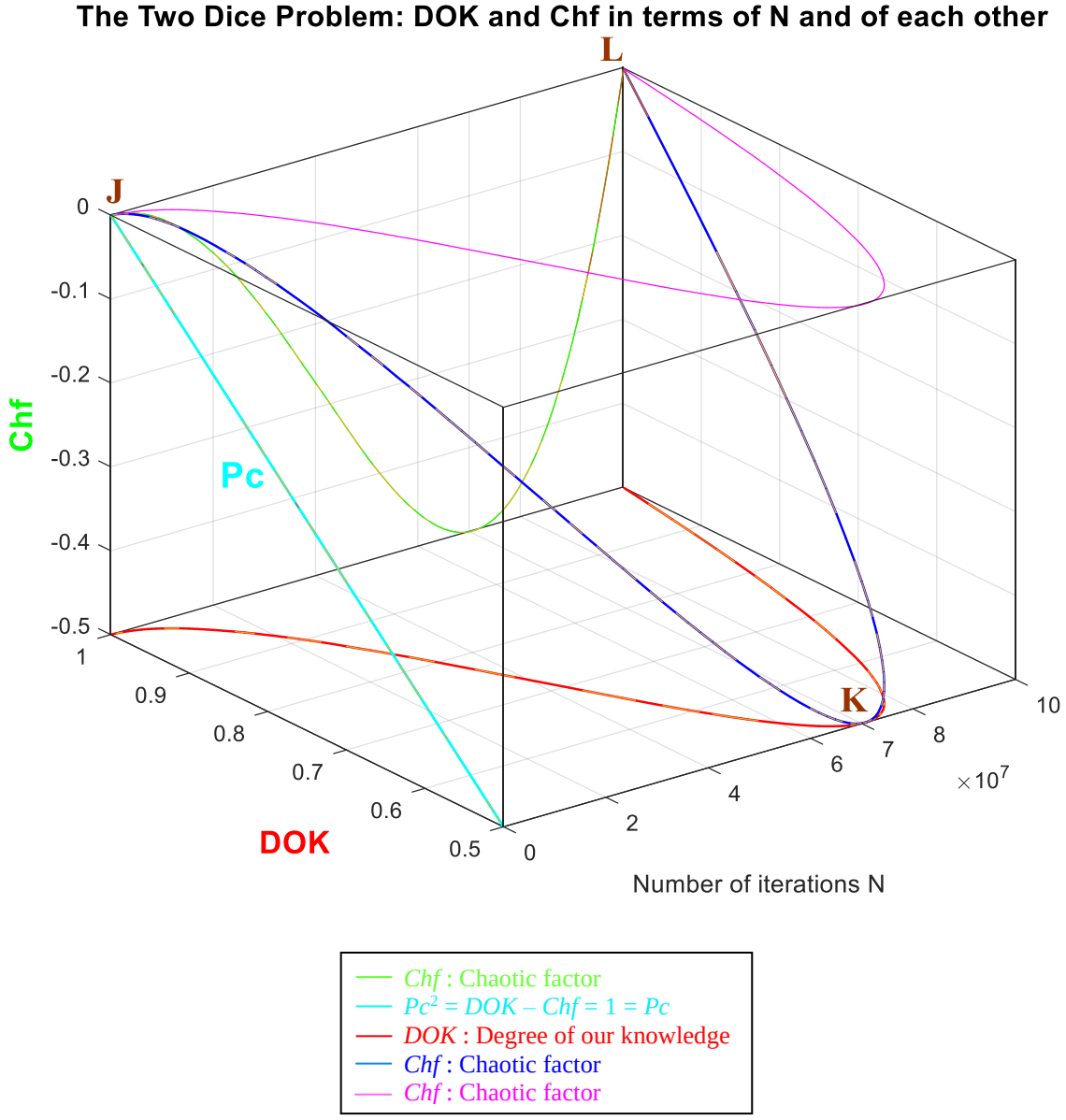


Figure 75. *DOK* and *Chf* in terms of *N* and of each other for the two dice problem.

In the second cube (Figure 76), we can notice the simulation of the convergence probability $P_r(N)$ and its complementary real divergence probability $P_m(N)/i$ in terms of the iterations N for the two dice problem. The line in cyan is the projection of $Pc^2(N) = P_r(N) + P_m(N)/i = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point ($P_r = 0$, $P_m/i = 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $P_r(N)$ in the plane $P_r(N) = P_m(N)/i$. This curve starts at the point J ($P_r = 0$, $P_m/i = 1$, $N = 0$ iterations), reaches the point K ($P_r = 0.5$, $P_m/i = 0.5$, $N =$ 70,000,000 iterations), and gets at the end to L ($P_r = 1$, $P_m/i = 0$, $N = N_C =$ 100,000,000 iterations). The blue curve represents $P_m(N)/i$ in the plane $P_r(N) + P_m(N)/i = 1$. Notice the importance of the point K which is the intersection of the red and blue curves at $N =$ 70,000,000 iterations and when $P_r(N) = P_m(N)/i = 0.5$. The three points J, K, L are the same as in Figure 74.

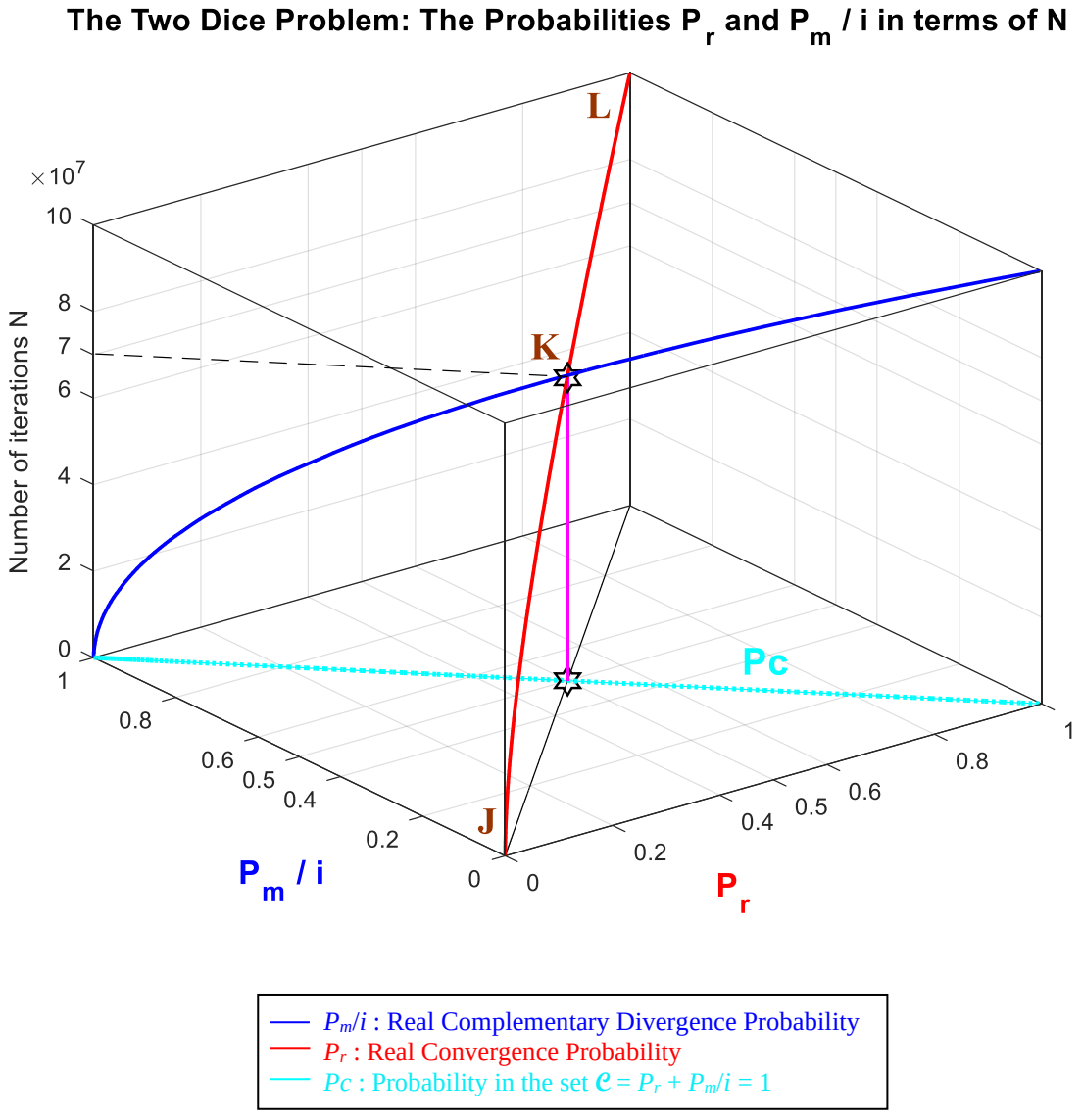


Figure 76. P_r and P_m/i in terms of N and of each other for the two dice problem.

In the third cube (Figure 77), we can notice the simulation of the complex random vector $Z(N)$ in $\mathcal{C}$ as a function of the real convergence probability $P_r(N) = \mathrm{Re}(Z)$ in $\mathcal{R}$ and of its complementary imaginary divergence probability $P_m(N) = i\times\mathrm{Im}(Z)$ in $\mathcal{M}$, and this in terms of the iterations N for the two dice problem. The red curve represents $P_r(N)$ in the plane $P_m(N) = 0$ and the blue curve represents $P_m(N)$ in the plane $P_r(N) = 0$. The green curve represents the complex probability vector $Z(N) = P_r(N) + P_m(N) = \mathrm{Re}(Z) + i\times\mathrm{Im}(Z)$ in the plane $P_r(N) = iP_m(N) + 1$. The curve of $Z(N)$ starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point L ($P_r = 1$, $P_m = 0$, $N = N_C = 100{,}000{,}000$ iterations). The line in cyan is $P_r(0) = iP_m(0) + 1$ and it is the projection of the $Z(N)$ curve on the complex probability plane whose equation is $N = 0$ iterations. This projected line starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point ($P_r = 1$, $P_m = 0$, $N = 0$ iterations). Notice the importance of the point K corresponding to $N = 70{,}000{,}000$ iterations and when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in Figure 74.

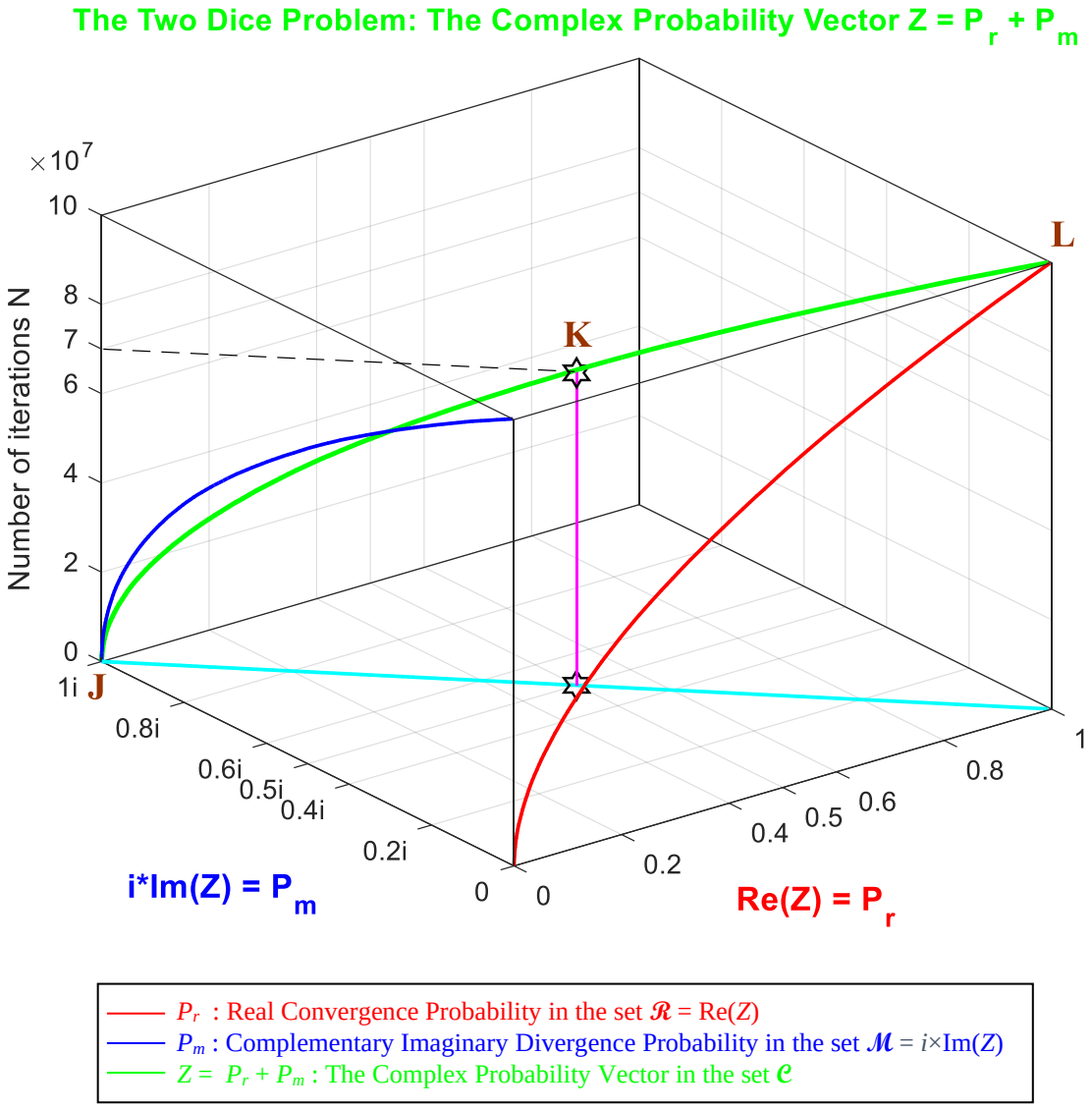


Figure 77. The Complex Probability Vector Z in terms of N for the two dice problem.

X- Conclusion and Perspectives

In the present research work the novel extended Kolmogorov paradigm of eight axioms (*EKA*) was applied and bonded to the classical and stochastic Monte Carlo numerical methods. Hence, a tight link between Monte Carlo methods and the original paradigm was made. Therefore, the model of "Complex Probability" was more elaborated beyond the scope of my previous research works on this subject.

Additionally, as it was verified and shown in the novel model, when $N = 0$ (before the beginning of the random simulation) and when $N = N_C$ (when Monte Carlo method converges to the exact result) therefore the degree of our knowledge (*DOK*) is one and the chaotic factor (*Chf* and *MChf*) is zero since the random effects and fluctuations have either not started or they have finished their task on the experiment. During the course of the stochastic experiment ($N > 0$) we have: $0.5 \leq DOK < 1$, $-0.5 \leq Chf < 0$, and $0 < MChf \leq 0.5$. Notice that during this whole process we have always $Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$, that means that the simulation which looked to be stochastic and random in the set $\mathcal{R}$ is now certain and deterministic in the set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after the addition of the contributions of $\mathcal{M}$ to the phenomenon occurring in $\mathcal{R}$ and thus after subtracting and eliminating the chaotic factor from the degree of our knowledge. Moreover, the convergence and divergence probabilities of the stochastic Monte Carlo method corresponding to each iteration cycle *N* have been evaluated in the probability sets $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$ by P_r, P_m, and *Pc* respectively. Consequently, at each instance of *N*, the new Monte Carlo method and *CPP* parameters R_E, R_A, P_r, P_m, P_m / i, *DOK*, *Chf*, *MChf*, *Pc*, and *Z* are certainly and perfectly predicted in the complex probability set $\mathcal{C}$ with *Pc* maintained as equal to one constantly and permanently.

In addition, using all these illustrated simulations and drawn graphs all over the whole research work, we can quantify and visualize both the certain knowledge (expressed by *DOK* and *Pc*) and the system chaos and random effects (expressed by *Chf* and *MChf*) of Monte Carlo methods. This is definitely very fascinating, fruitful, and wonderful and proves once again the advantages of extending the five probability axioms of Kolmogorov and thus the novelty and benefits of this original field in prognostic and applied mathematics that can be called verily:

"The Complex Probability Paradigm".

Furthermore, it is important to indicate here that one very well-known and essential probability distribution was considered in the present paper which is the discrete uniform probability distribution as well as a specific uniform random numbers generator, knowing that the novel *CPP* model can be applied to any uniform random numbers' generator existent in literature. This will lead certainly to analogous conclusions and results and will show undoubtedly the success of my original theory.

Moreover, it is also significant to mention that it is possible to compare the current conclusions and results with the existing ones from both theoretical investigations and analysis and simulation researches and studies. This will be the task of subsequent research papers.

As a prospective and future work and challenges, it is planned to more elaborate the original created prognostic paradigm and to implement it to a varied set of nondeterministic systems like

for other random experiments in classical probability theory and in stochastic processes. Furthermore, we will apply also *CPP* to the field of prognostic in engineering as well as to other random problems which have enormous applications in physics, in economics, in chemistry, in applied and pure mathematics.

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this research work.

CHAPTER TWO

THE MONTÉ CARLO TECHNIQUES AND THE COMPLEX PROBABILITY PARADIGM

"Chance is the pseudonym of God when He did not want to sign."
Anatole France.

"There is a certain Eternal Law, to wit, Reason, existing in the mind of God and governing the whole universe."
Saint Thomas Aquinas.

"An equation has no meaning for me unless it expresses a thought of God."
Srinivasa Ramanujan.

Abstract: The five fundamental axioms of classical probability theory were put forward in 1933 by *Andrey Nikolaevich Kolmogorov*. Encompassing new imaginary dimensions with the experiment real dimensions will make the work in the complex probability set $\mathcal{C}$ totally predictable and with a probability permanently equal to one. This is the original idea in my complex probability paradigm. Therefore, this will make the event in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ absolutely deterministic by adding to the real set of probabilities $\mathcal{R}$ the contributions of the imaginary set of probabilities $\mathcal{M}$. It is of great importance that stochastic systems become totally predictable since we will be perfectly knowledgeable to foretell the outcome of all random events that occur in nature. Consequently, by calculating the parameters of the new prognostic model, we will be able to determine the chaotic factor, the magnitude of the chaotic factor, the degree of our knowledge, the real and imaginary and complex probabilities in the probability sets $\mathcal{R}$ and $\mathcal{M}$ and $\mathcal{C}$ and which are all subject to chaos and random effects. We will apply this innovative paradigm to the well-known Monte Carlo techniques and to their random algorithms and procedures in a novel way.

Keywords: Degree of our knowledge, chaotic factor, complex probability set, probability norm, complex random vector, convergence probability, divergence probability, simulation.

NOMENCLATURE

$\mathcal{R}$ = the events real set
$\mathcal{M}$ = the events imaginary set
$\mathcal{C}$ = the events complex set
i = the imaginary number with $i^2 = -1$ or $i = \sqrt{-1}$
EKA = Extended Kolmogorov's Axioms
CPP = Complex Probability Paradigm
P_{rob} = any event probability

P_r = the probability in the real set $\mathcal{R}$ = convergence probability in $\mathcal{R}$

P_m = the probability in the complementary imaginary set $\mathcal{M}$ that corresponds to the real probability set in $\mathcal{R}$ = divergence probability in $\mathcal{M}$

Pc = the event probability in $\mathcal{R}$ with its associated event in $\mathcal{M}$ = probability in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$

R_E = the random experiment exact result

R_A = the random experiment approximate result

Z = complex probability number = complex random vector = sum of P_r and P_m

DOK = $|Z|^2$ = the degree of our knowledge of the stochastic experiment or system, it is the square of the norm of Z

Chf = the chaotic factor of Z

$MChf$ = the magnitude of the chaotic factor of Z

N = the number of iterations cycles = number of random vectors

N_C = the number of iterations cycles till the convergence of Monte Carlo method to R_E = the number of random vectors till convergence.

I- Introduction [1-90]

Computing probabilities is the main work of classical probability theory. Adding new dimensions to the stochastic experiments will lead to a deterministic expression of probability theory. This is the original idea at the foundations of this work. Actually, the theory of probability is a nondeterministic system in its essence; that means that the events outcomes are due to chance and randomness. The addition of novel imaginary dimensions to the chaotic experiment occurring in the set $\mathcal{R}$ will yield a deterministic experiment and hence a stochastic event will have a certain result in the complex probability set $\mathcal{C}$. If the random event becomes completely predictable then we will be fully knowledgeable to predict the outcome of stochastic experiments that arise in the real world in all stochastic processes. Consequently, the work that has been accomplished here was to extend the real probabilities set $\mathcal{R}$ to the deterministic complex probabilities set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ by including the contributions of the set $\mathcal{M}$ which is the imaginary set of probabilities. Therefore, since this extension was found to be successful, then a novel paradigm of stochastic sciences and prognostic and physics was laid down in which all stochastic phenomena in $\mathcal{R}$ was expressed deterministically. I coined this novel model by the term "The Complex Probability Paradigm" that was initiated and established in my earlier research works.

II- The Purpose and the Advantages of the Current Chapter [37-90]

The advantages and the purpose of the present chapter are to:

1- Extend the theory of classical probability to cover the complex numbers set, hence to connect the probability theory to the field of complex variables and analysis. This task was started and elaborated in my earlier works.
2- Apply the novel probability axioms and paradigm to Monte Carlo techniques.

3- Show that all nondeterministic phenomena can be expressed deterministically in the complex probabilities set which is $\mathcal{C}$.

4- Compute and quantify both the degree of our knowledge and the chaotic factor of Monte Carlo procedures.

5- Represent and show the graphs of the functions and parameters of the innovative paradigm related to Monte Carlo algorithms.

6- Demonstrate that the classical concept of probability is permanently equal to one in the set of complex probabilities; hence, no chaos, no randomness, no ignorance, no uncertainty, no unpredictability, no nondeterminism, and no disorder exist in:

$$\mathcal{C} \text{ (complex set)} = \mathcal{R} \text{ (real set)} + \mathcal{M} \text{ (imaginary set)}.$$

7- Pave the way to implement this inventive model to other topics in prognostics and to the field of stochastic processes. These will be the goals of my future research works.

Concerning some applications of the novel established paradigm and as a future work, it can be applied to any nondeterministic phenomena using Monte Carlo algorithms whether in the continuous or in the discrete cases. Moreover, compared with existing literature, the major contribution of the current research chapter is to apply the innovative paradigm of complex probability to the concepts and techniques of the probabilistic Monte Carlo simulations and algorithms. The next figure displays the major aims of the Complex Probability Paradigm (*CPP*) (Figure 1).

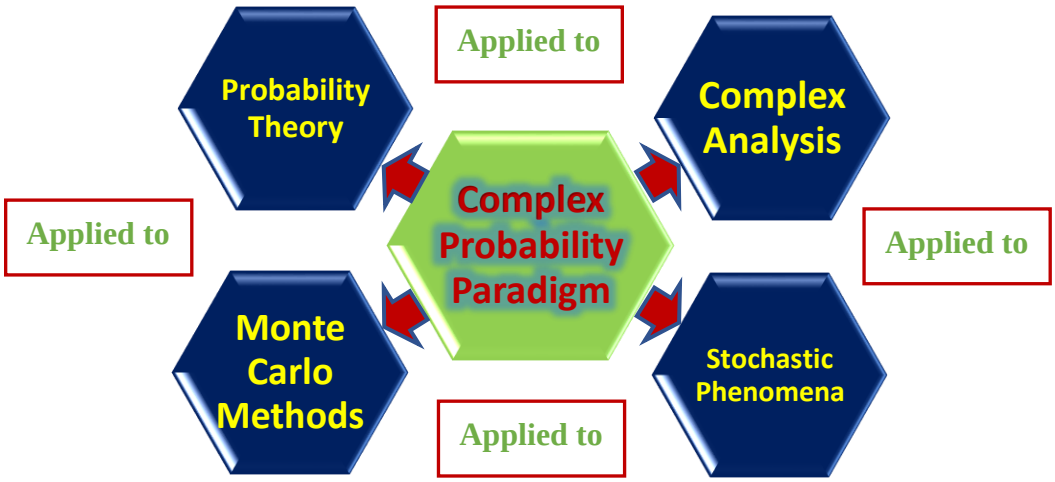


Figure 1. The diagram of the major aims of the Complex Probability Paradigm

III- The Complex Probability Paradigm [37-141]

III-1- The Original Andrey Nikolaevich Kolmogorov System of Axioms

The simplicity of Kolmogorov's system of axioms may be surprising. Let E be a collection of elements $\{E_1, E_2, \ldots\}$ called elementary events and let F be a set of subsets of E called random events. The five axioms for a finite set E are:

Axiom 1: F is a field of sets.
Axiom 2: F contains the set E.
Axiom 3: A non-negative real number $P_{rob}(A)$, called the probability of A, is assigned to each set A in F. We have always $0 \le P_{rob}(A) \le 1$.
Axiom 4: $P_{rob}(E)$ equals 1.
Axiom 5: If A and B have no elements in common, the number assigned to their union is:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B)$$

hence, we say that A and B are disjoint; otherwise, we have:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B) - P_{rob}(A \cap B)$$

And we say also that: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B/A) = P_{rob}(B) \times P_{rob}(A/B)$ which is the conditional probability. If both A and B are independent then: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B)$.

Moreover, we can generalize and say that for N disjoint (mutually exclusive) events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following additivity rule:

$$P_{rob}\left(\bigcup_{j=1}^{N} A_j\right) = \sum_{j=1}^{N} P_{rob}\left(A_j\right)$$

And we say also that for N independent events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following product rule:

$$P_{rob}\left(\bigcap_{j=1}^{N} A_j\right) = \prod_{j=1}^{N} P_{rob}\left(A_j\right)$$

III-2- Adding the Imaginary Part $\mathcal{M}$

Now, we can add to this system of axioms an imaginary part such that:

Axiom 6: Let $P_m = i \times (1 - P_r)$ be the probability of an associated complementary event in $\mathcal{M}$ (the imaginary part) to the event A in $\mathcal{R}$ (the real part). It follows that $P_r + P_m / i = 1$ where i is the imaginary number with $i = \sqrt{-1}$ or $i^2 = -1$.

Axiom 7: We construct the complex number or vector $Z = P_r + P_m = P_r + i(1 - P_r)$ having a norm $|Z|$ such that:

$$|Z|^2 = P_r^2 + (P_m / i)^2 .$$

Axiom 8: Let *Pc* denote the probability of an event in the complex probability universe $\mathcal{C}$ where $\mathcal{C} = \mathcal{R} + \mathcal{M}$. We say that *Pc* is the probability of an event *A* in $\mathcal{R}$ with its associated event in $\mathcal{M}$ such that:

$$Pc^2 = (P_r + P_m / i)^2 = |Z|^2 - 2iP_rP_m \quad \text{and is always equal to 1.}$$

We can see that by taking into consideration the set of imaginary probabilities we added three new and original axioms and consequently the system of axioms defined by Kolmogorov was hence expanded to encompass the set of imaginary numbers.

III-3- A Brief Interpretation of the Novel Paradigm

To conclude and to summarize my original invented model, as the degree of our certain knowledge *DOK* in the real probability universe and set $\mathcal{R}$ is unfortunately imperfect and incomplete and hence unsatisfactory, then the extension to the complex probability set $\mathcal{C}$ includes the contributions of both the real set of probabilities $\mathcal{R}$ and the imaginary set of probabilities $\mathcal{M}$. Consequently, this will result to a complete and perfect degree of knowledge in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ since *Pc* = 1 constantly and permanently. In fact, in order to have a certain prediction of any random event, it is necessary to work in the complex set $\mathcal{C}$ in which the chaotic factor *Chf* is quantified and subtracted from the computed degree of knowledge to lead to a probability in $\mathcal{C}$ always equal to one as it is proved and shown in the following equation:

$$Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$$

and which was derived from the Complex Probability Paradigm. This hypothesis and innovative and original model are verified by the mean of many examples encompassing both various, important, and well-known discrete and continuous probability distributions illustrated and discussed in my previous research works.

The figure that follows shows and summarizes the Extended Kolmogorov Axioms (*EKA*) or the Complex Probability Paradigm (*CPP*) (Figure 2):

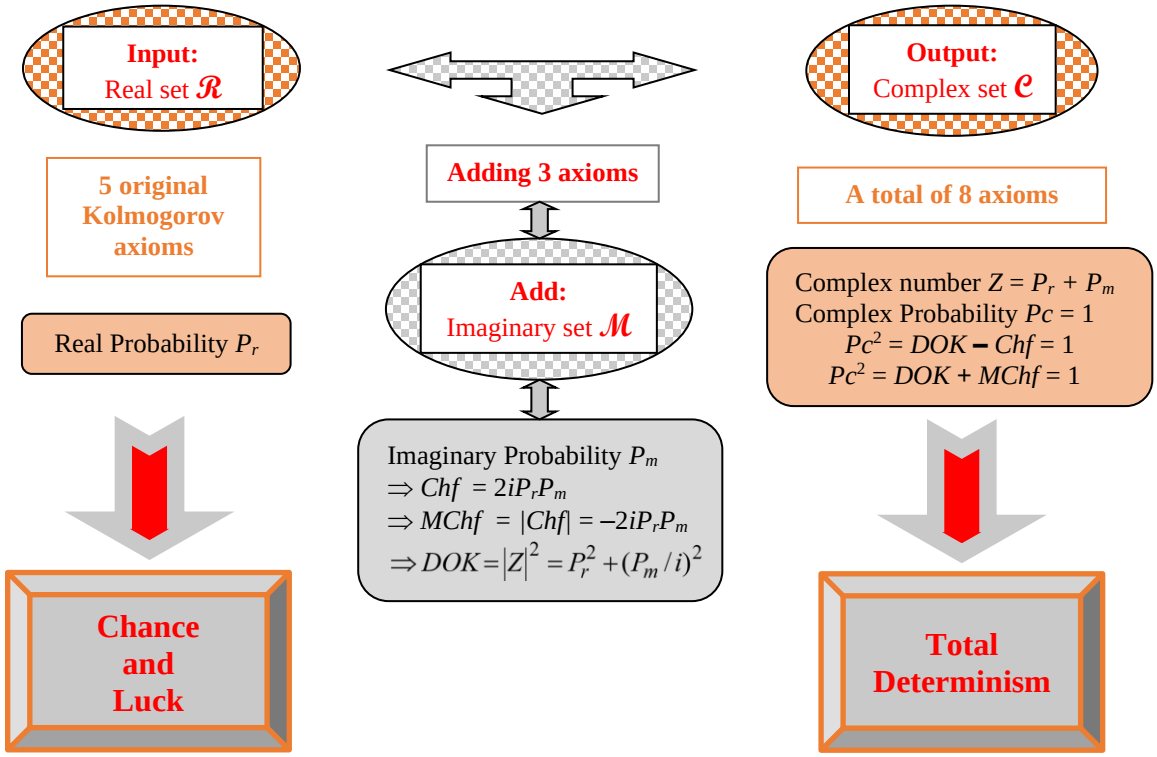


Figure 2- The *EKA* or the *CPP* diagram

IV- The Monte Carlo Techniques and the Complex Probability Paradigm Parameters [37-141]

IV-1- The Convergence and Divergence Probabilities

Let R_E be the exact result of the random experiment or of a simple or a multidimensional integral that are not always possible to evaluate by ordinary methods of probability theory or calculus or deterministic numerical methods. And let R_A be the approximate result of these experiments and integrals found by Monte Carlo techniques.

The relative error in the Monte Carlo methods is: Rel. Error $= \left|\frac{R_E - R_A}{R_E}\right| = \left|1 - \frac{R_A}{R_E}\right|$

In addition, the percent relative error is $= 100\% \times \left|\frac{R_E - R_A}{R_E}\right|$ and is always between 0% and 100%.

Therefore, the relative error is always between 0 and 1. Hence:

$$0 \le \left| \frac{R_E - R_A}{R_E} \right| \le 1 \Leftrightarrow \begin{cases} 0 \le \left(\frac{R_E - R_A}{R_E} \right) \le 1 & \text{if } R_A \le R_E \\ 0 \le -\left(\frac{R_E - R_A}{R_E} \right) \le 1 & \text{if } R_A \ge R_E \end{cases} \Leftrightarrow \begin{cases} 0 \le R_A \le R_E \\ R_E \le R_A \le 2R_E \end{cases}$$

Moreover, we define the real probability in the set $\mathcal{R}$ by:

$$P_r = 1 - \left| \frac{R_E - R_A}{R_E} \right| = 1 - \left| 1 - \frac{R_A}{R_E} \right| = \begin{cases} 1 - \left(1 - \frac{R_A}{R_E} \right) & \text{if } 0 \le R_A \le R_E \\ 1 + \left(1 - \frac{R_A}{R_E} \right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = \begin{cases} \frac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ 2 - \frac{R_A}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

= 1 – the relative error in the Monte Carlo method
= Probability of Monte Carlo method convergence in $\mathcal{R}$.

And therefore:

$$P_m = i\left(1 - P_r\right) = i\left\{1 - \left[1 - \left| \frac{R_E - R_A}{R_E} \right|\right]\right\} = i\left\{1 - \left[1 - \left|1 - \frac{R_A}{R_E}\right|\right]\right\} = i\left|1 - \frac{R_A}{R_E}\right|$$

$$= \begin{cases} i\left(1 - \frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ -i\left(1 - \frac{R_A}{R_E}\right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = \begin{cases} i\left(1 - \frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ i\left(\frac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

= Probability of Monte Carlo method divergence in the imaginary complementary probability set $\mathcal{M}$ since it is the imaginary complement of P_r.
Consequently,

$$P_m / i = 1 - P_r = \left|1 - \frac{R_A}{R_E}\right| = \begin{cases} 1 - \frac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ \frac{R_A}{R_E} - 1 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

= The relative error in the Monte Carlo method
= Probability of Monte Carlo method divergence in $\mathcal{R}$ since it is the real complement of P_r.

In the case where $0 \le R_A \le R_E \Rightarrow 0 \le \frac{R_A}{R_E} \le 1 \Rightarrow 0 \le P_r \le 1$ and we deduce also that $0 \le \left(1 - \frac{R_A}{R_E}\right) \le 1$
$\Rightarrow 0 \le P_m / i \le 1$ and $\Rightarrow 0 \le P_m \le i$

And in the case where $R_E \le R_A \le 2R_E \Rightarrow 1 \le \frac{R_A}{R_E} \le 2 \Rightarrow 0 \le \left(2 - \frac{R_A}{R_E}\right) \le 1 \Rightarrow 0 \le P_r \le 1$ and we deduce also that $0 \le \left(\frac{R_A}{R_E} - 1\right) \le 1 \Rightarrow 0 \le P_m / i \le 1$ and $\Rightarrow 0 \le P_m \le i$

Therefore, if $R_A = 0$ or $R_A = 2R_E$ that means before the beginning of the simulation, then:

$$P_r = P_{rob}\,(\text{convergence}) \text{ in } \mathcal{R} = 0$$
$$P_m = P_{rob}\,(\text{divergence}) \text{ in } \mathcal{M} = i$$
$$P_m / i = P_{rob}\,(\text{divergence}) \text{ in } \mathcal{R} = 1$$

And if $R_A = R_E$ that means at the end of Monte Carlo simulation then:

$$P_r = P_{rob}\,(\text{convergence}) \text{ in } \mathcal{R} = 1$$
$$P_m = P_{rob}\,(\text{divergence}) \text{ in } \mathcal{M} = 0$$
$$P_m / i = P_{rob}\,(\text{divergence}) \text{ in } \mathcal{R} = 0$$

IV-2- The Complex Random Vector *Z* in $\mathcal{C} = \mathcal{R} + \mathcal{M}$

We have $Z = P_r + P_m = \begin{cases} \frac{R_A}{R_E} + i\left(1 - \frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ \left(2 - \frac{R_A}{R_E}\right) + i\left(\frac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = \mathrm{Re}(Z) + i\,\mathrm{Im}(Z)$

where $\mathrm{Re}(Z) = P_r = \begin{cases} \frac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ 2 - \frac{R_A}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases}$ = the real part of *Z*

and $\mathrm{Im}(Z) = P_m / i = \begin{cases} 1 - \frac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ \frac{R_A}{R_E} - 1 & \text{if } R_E \le R_A \le 2R_E \end{cases}$ = the imaginary part of *Z*.

That means that the complex random vector *Z* is the sum in $\mathcal{C}$ of the real probability of convergence in $\mathcal{R}$ and of the imaginary probability of divergence in $\mathcal{M}$.

If $R_A = 0$ or $R_A = 2R_E$ (before the simulation begins) then:

$P_r = \frac{R_A}{R_E} = \frac{0}{R_E} = 0$ or $P_r = 2 - \frac{R_A}{R_E} = 2 - \frac{2R_E}{R_E} = 2 - 2 = 0$

and $P_m = i\left(1-\frac{R_A}{R_E}\right) = i\left(1-\frac{0}{R_E}\right) = i(1-0) = i$ or $P_m = i\left(\frac{R_A}{R_E}-1\right) = i\left(\frac{2R_E}{R_E}-1\right) = i(2-1) = i$

therefore $Z = 0+i = i$.

If $R_A = \frac{R_E}{2}$ or $R_A = \frac{3R_E}{2}$ (at the middle of the simulation) then:

$$P_r = \begin{cases} \frac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ 2-\frac{R_A}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases} = \begin{cases} \frac{R_E}{2R_E} = 0.5 & \text{if } 0 \le R_A \le R_E \\ 2-\frac{3R_E}{2R_E} = 0.5 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow P_r = 0.5$$

and $$P_m = \begin{cases} i\left(1-\frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ i\left(\frac{R_A}{R_E}-1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = \begin{cases} i\left(1-\frac{R_E}{2R_E}\right) = 0.5i & \text{if } 0 \le R_A \le R_E \\ i\left(\frac{3R_E}{2R_E}-1\right) = 0.5i & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow P_m = 0.5i$$

therefore $Z = 0.5+0.5i$.

If $R_A = R_E$ (at the simulation end) then:

$$P_r = \begin{cases} \frac{R_A}{R_E} = \frac{R_E}{R_E} = 1 & \text{if } 0 \le R_A \le R_E \\ 2-\frac{R_A}{R_E} = 2-\frac{R_E}{R_E} = 2-1 = 1 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow P_r = 1$$

And

$$P_m = \begin{cases} i\left(1-\frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ i\left(\frac{R_A}{R_E}-1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = \begin{cases} i\left(1-\frac{R_E}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ i\left(\frac{R_E}{R_E}-1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = \begin{cases} 0 & \text{if } 0 \le R_A \le R_E \\ 0 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$\Leftrightarrow P_m = 0$

therefore $Z = 1+0i = 1$.

IV-3- The Degree of Our Knowledge *DOK*

We have:

$$DOK = |Z|^2 = P_r^2 + (P_m / i)^2 = \begin{cases} \left(\frac{R_A}{R_E}\right)^2 & \text{if } 0 \le R_A \le R_E \\ \left(2 - \frac{R_A}{R_E}\right)^2 & \text{if } R_E \le R_A \le 2R_E \end{cases} + \begin{cases} \left(1 - \frac{R_A}{R_E}\right)^2 & \text{if } 0 \le R_A \le R_E \\ \left(\frac{R_A}{R_E} - 1\right)^2 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$= \begin{cases} \left(\frac{R_A}{R_E}\right)^2 + \left(1 - \frac{R_A}{R_E}\right)^2 & \text{if } 0 \le R_A \le R_E \\ \left(2 - \frac{R_A}{R_E}\right)^2 + \left(\frac{R_A}{R_E} - 1\right)^2 & \text{if } R_E \le R_A \le 2R_E \end{cases} = \begin{cases} 2\left(\frac{R_A}{R_E}\right)^2 - 2\left(\frac{R_A}{R_E}\right) + 1 & \text{if } 0 \le R_A \le R_E \\ 2\left(\frac{R_A}{R_E}\right)^2 - 6\left(\frac{R_A}{R_E}\right) + 5 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

From *CPP* we have that $0.5 \le DOK \le 1$ then if $DOK = 0.5$

$$\Leftrightarrow \begin{cases} 2\left(\frac{R_A}{R_E}\right)^2 - 2\left(\frac{R_A}{R_E}\right) + 1 = 0.5 & \text{if } 0 \le R_A \le R_E \\ 2\left(\frac{R_A}{R_E}\right)^2 - 6\left(\frac{R_A}{R_E}\right) + 5 = 0.5 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

then solving the second-degree equations for $\frac{R_A}{R_E}$ gives:

$$\begin{cases} \frac{R_A}{R_E} = 1/2 & \text{if } 0 \le R_A \le R_E \\ \frac{R_A}{R_E} = 3/2 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} R_A = R_E / 2 & \text{if } 0 \le R_A \le R_E \\ R_A = 3R_E / 2 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$ and vice versa.

That means that *DOK* is minimum when the approximate result R_A is equal to half of the exact result R_E if $0 \le R_A \le R_E$ or when the approximate result is equal to three times the half of the exact result if $R_E \le R_A \le 2R_E$, that means at the middle of the simulation.

In addition, if $DOK = 1$ then:

$$\Leftrightarrow \begin{cases} 2\left(\frac{R_A}{R_E}\right)^2 - 2\left(\frac{R_A}{R_E}\right) + 1 = 1 & \text{if } 0 \le R_A \le R_E \\ 2\left(\frac{R_A}{R_E}\right)^2 - 6\left(\frac{R_A}{R_E}\right) + 5 = 1 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} \left(\frac{R_A}{R_E}\right)^2 - \left(\frac{R_A}{R_E}\right) = 0 & \text{if } 0 \le R_A \le R_E \\ 2\left(\frac{R_A}{R_E}\right)^2 - 6\left(\frac{R_A}{R_E}\right) + 4 = 0 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$\Leftrightarrow \begin{cases} R_A = 0 \text{ OR } R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = 2R_E \text{ OR } R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases}$$ and vice versa.

That means that *DOK* is maximum when the approximate result R_A is equal to 0 or $2R_E$ (before the beginning of the simulation) and when it is equal to the exact result R_E (at the end of the simulation). We can deduce that we have perfect and total knowledge of the stochastic experiment before the beginning of Monte Carlo simulation since no randomness was introduced yet, as well as at the end of the simulation after the convergence of the method to the exact result.

IV-4- The Chaotic Factor *Chf*

We have:

$$Chf = 2iP_rP_m = 2i \times \begin{cases} \dfrac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ 2 - \dfrac{R_A}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases} \times \begin{cases} i\left(1 - \dfrac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ i\left(\dfrac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

Since $i^2 = -1$ then:

$$Chf = \begin{cases} -2\left(\dfrac{R_A}{R_E}\right)\left(1 - \dfrac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ -2\left(2 - \dfrac{R_A}{R_E}\right)\left(\dfrac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

From *CPP* we have that $-0.5 \le Chf \le 0$ then if $Chf = -0.5$

$$\Leftrightarrow \begin{cases} -2\left(\dfrac{R_A}{R_E}\right)\left(1 - \dfrac{R_A}{R_E}\right) = -0.5 & \text{if } 0 \le R_A \le R_E \\ -2\left(2 - \dfrac{R_A}{R_E}\right)\left(\dfrac{R_A}{R_E} - 1\right) = -0.5 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} R_A = R_E / 2 & \text{if } 0 \le R_A \le R_E \\ R_A = 3R_E / 2 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

and vice versa.

That means that *Chf* is minimum when the approximate result R_A is equal to half of the exact result R_E if $0 \le R_A \le R_E$ or when the approximate result is equal to three times the half of the exact result if $R_E \le R_A \le 2R_E$, that means at the middle of the simulation.
In addition, if $Chf = 0$ then:

$$\Leftrightarrow \begin{cases} -2\left(\dfrac{R_A}{R_E}\right)\left(1 - \dfrac{R_A}{R_E}\right) = 0 & \text{if } 0 \le R_A \le R_E \\ -2\left(2 - \dfrac{R_A}{R_E}\right)\left(\dfrac{R_A}{R_E} - 1\right) = 0 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} R_A = 0 \text{ OR } R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = 2R_E \text{ OR } R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

And, conversely, if $\begin{cases} R_A = 0 \text{ OR } R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = 2R_E \text{ OR } R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases}$ then $Chf = 0$.

That means that *Chf* is equal to 0 when the approximate result R_A is equal to 0 or $2R_E$ (before the beginning of the simulation) and when it is equal to the exact result R_E (at the end of the simulation).

IV-5- The Magnitude of the Chaotic Factor *MChf*

We have:

$$MChf = |Chf| = -2iP_rP_m = -2i \times \begin{cases} \dfrac{R_A}{R_E} & \text{if } 0 \le R_A \le R_E \\ 2 - \dfrac{R_A}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases} \times \begin{cases} i\left(1 - \dfrac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ i\left(\dfrac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

Since $i^2 = -1$ then:

$$MChf = \begin{cases} 2\left(\dfrac{R_A}{R_E}\right)\left(1 - \dfrac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ 2\left(2 - \dfrac{R_A}{R_E}\right)\left(\dfrac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

From *CPP* we have that $0 \le MChf \le 0.5$ then if $MChf = 0.5$

$$\Leftrightarrow \begin{cases} 2\left(\dfrac{R_A}{R_E}\right)\left(1 - \dfrac{R_A}{R_E}\right) = 0.5 & \text{if } 0 \le R_A \le R_E \\ 2\left(2 - \dfrac{R_A}{R_E}\right)\left(\dfrac{R_A}{R_E} - 1\right) = 0.5 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} R_A = R_E / 2 & \text{if } 0 \le R_A \le R_E \\ R_A = 3R_E / 2 & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

and vice versa.

That means that *MChf* is maximum when the approximate result R_A is equal to half of the exact result R_E if $0 \le R_A \le R_E$ or when the approximate result is equal to three times the half of the exact result if $R_E \le R_A \le 2R_E$, that means at the middle of the simulation. This implies that the magnitude of the chaos (*MChf*) introduced by the random variables used in Monte Carlo method is maximum at the halfway of the simulation.

In addition, if $MChf = 0$ then:

$$\Leftrightarrow \begin{cases} 2\left(\frac{R_A}{R_E}\right)\left(1-\frac{R_A}{R_E}\right)=0 & \text{if } 0 \le R_A \le R_E \\ 2\left(2-\frac{R_A}{R_E}\right)\left(\frac{R_A}{R_E}-1\right)=0 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} R_A = 0 \text{ OR } R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = 2R_E \text{ OR } R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

And, conversely, if $\Leftrightarrow \begin{cases} R_A = 0 \text{ OR } R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = 2R_E \text{ OR } R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases}$ then $MChf = 0$.

That means that *MChf* is minimum and is equal to 0 when the approximate result R_A is equal to 0 or $2R_E$ (before the beginning of the simulation) and when it is equal to the exact result R_E (at the end of the simulation). We can deduce that the magnitude of the chaos in the stochastic experiment is null before the beginning of Monte Carlo simulation since no randomness was introduced yet, as well as at the end of the simulation after the convergence of the method to the exact result when randomness has finished its task in the stochastic Monte Carlo method and experiment.

IV-6- The Probability *Pc* in the Probability Set $\mathcal{C} = \mathcal{R} + \mathcal{M}$

We have:

$$Pc^2 = DOK - Chf = DOK + MChf$$

$$= \begin{cases} 2\left(\frac{R_A}{R_E}\right)^2 - 2\left(\frac{R_A}{R_E}\right)+1 & \text{if } 0 \le R_A \le R_E \\ 2\left(\frac{R_A}{R_E}\right)^2 - 6\left(\frac{R_A}{R_E}\right)+5 & \text{if } R_E \le R_A \le 2R_E \end{cases} - \begin{cases} -2\left(\frac{R_A}{R_E}\right)\left(1-\frac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ -2\left(2-\frac{R_A}{R_E}\right)\left(\frac{R_A}{R_E}-1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$= \begin{cases} 1 & \text{if } 0 \le R_A \le R_E \\ 1 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow Pc^2 = 1 \quad \text{for } 0 \le \forall R_A \le 2R_E$$

$\Leftrightarrow Pc = 1$ = Probability of convergence in $\mathcal{C}$, therefore:

$$Pc = \begin{cases} \frac{R_A}{R_E} = 1 & \text{if } 0 \le R_A \le R_E \\ 2-\frac{R_A}{R_E} = 1 & \text{if } R_E \le R_A \le 2R_E \end{cases} \Leftrightarrow \begin{cases} R_A = R_E & \text{if } 0 \le R_A \le R_E \\ R_A = R_E & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$\Leftrightarrow R_A = R_E$ for $0 \le \forall R_A \le 2R_E$ continuously in the probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$. This is due to the fact that in $\mathcal{C}$ we have subtracted in the equation above the chaotic factor *Chf* from our knowledge *DOK* and therefore we have eliminated chaos caused and introduced by all the random variables and the stochastic fluctuations that lead to approximate results in the Monte Carlo simulation in $\mathcal{R}$. Therefore, since in $\mathcal{C}$ we have always $R_A = R_E$ then the Monte Carlo simulation which is a stochastic method by nature in $\mathcal{R}$ becomes after applying the *CPP* a deterministic method in $\mathcal{C}$

since the probability of convergence of any random experiment in $\mathcal{C}$ is constantly and permanently equal to 1 for any iterations number N.

IV-7- The Rates of Change of the Probabilities in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$

$$\text{Since } Z = P_r + P_m = \begin{cases} \dfrac{R_A}{R_E} + i\left(1 - \dfrac{R_A}{R_E}\right) & \text{if } 0 \le R_A \le R_E \\ \left(2 - \dfrac{R_A}{R_E}\right) + i\left(\dfrac{R_A}{R_E} - 1\right) & \text{if } R_E \le R_A \le 2R_E \end{cases} = \mathrm{Re}(Z) + i\,\mathrm{Im}(Z)$$

Then:

$$\frac{dZ}{dR_A} = \frac{dP_r}{dR_A} + \frac{dP_m}{dR_A} = \begin{cases} \dfrac{d}{dR_A}\left[\dfrac{R_A}{R_E} + i\left(1 - \dfrac{R_A}{R_E}\right)\right] & \text{if } 0 \le R_A \le R_E \\ \dfrac{d}{dR_A}\left[\left(2 - \dfrac{R_A}{R_E}\right) + i\left(\dfrac{R_A}{R_E} - 1\right)\right] & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$= \begin{cases} \dfrac{d}{dR_A}\left[\dfrac{R_A}{R_E}\right] + \dfrac{d}{dR_A}\left[i\left(1 - \dfrac{R_A}{R_E}\right)\right] & \text{if } 0 \le R_A \le R_E \\ \dfrac{d}{dR_A}\left[2 - \dfrac{R_A}{R_E}\right] + \dfrac{d}{dR_A}\left[i\left(\dfrac{R_A}{R_E} - 1\right)\right] & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$= \begin{cases} \dfrac{1}{R_E} - \dfrac{i}{R_E} = \dfrac{1}{R_E}(1 - i) & \text{if } 0 \le R_A \le R_E \\ -\dfrac{1}{R_E} + \dfrac{i}{R_E} = \dfrac{1}{R_E}(i - 1) & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

Therefore,

- $$\mathrm{Re}\left[\frac{dZ}{dR_A}\right] = \frac{dP_r}{dR_A} = \begin{cases} +\dfrac{1}{R_E} & \text{if } 0 \le R_A \le R_E \\ -\dfrac{1}{R_E} & \text{if } R_E \le R_A \le 2R_E \end{cases}$$

$$= \begin{cases} \text{constant} > 0 & \text{if } 0 \le R_A \le R_E \text{ and } R_E > 0 \\ \text{constant} < 0 & \text{if } R_E \le R_A \le 2R_E \text{ and } R_E > 0 \end{cases}$$

that means that the slope of the probability of convergence in $\mathcal{R}$ or its rate of change is constant and positive if $0 \le R_A \le R_E$, and constant and negative if $R_E \le R_A \le 2R_E$, and it depends only on R_E ; hence, we have a constant increase in P_r (the convergence probability) as a function of the iterations number N as R_A increases from 0 to R_E and as

R_A decreases from $2R_E$ to R_E till P_r reaches the value 1 that means till the random experiment converges to R_E.

- $$\mathrm{Im}\left[\frac{dZ}{dR_A}\right]=\frac{1}{i}\frac{dP_m}{dR_A}=\frac{d(P_m/i)}{dR_A}=\begin{cases}-\dfrac{1}{R_E} & \text{if } 0\le R_A\le R_E\\ +\dfrac{1}{R_E} & \text{if } R_E\le R_A\le 2R_E\end{cases}$$
$$=\begin{cases}\text{constant}<0 & \text{if } 0\le R_A\le R_E \text{ and } R_E>0\\ \text{constant}>0 & \text{if } R_E\le R_A\le 2R_E \text{ and } R_E>0\end{cases}$$
that means that the slopes of the probabilities of divergence in $\mathcal{R}$ and $\mathcal{M}$ or their rates of change are constant and negative if $0\le R_A\le R_E$, and constant and positive if $R_E\le R_A\le 2R_E$, and they depend only on R_E; hence, we have a constant decrease in P_m/i and P_m (the divergence probabilities) as functions of the iterations number *N* as R_A increases from 0 to R_E and as R_A decreases from $2R_E$ to R_E till P_m/i and P_m reach the value 0 that means till the random experiment converges to R_E.

Additionally,

$$\left|\frac{dZ}{dR_A}\right|^2=\left[\frac{dP_r}{dR_A}\right]^2+\left[\frac{1}{i}\frac{dP_m}{dR_A}\right]^2=\left[\frac{dP_r}{dR_A}\right]^2+\left[\frac{d(P_m/i)}{dR_A}\right]^2$$
$$=\begin{cases}\left(\dfrac{1}{R_E}\right)^2+\left(-\dfrac{1}{R_E}\right)^2 & \text{if } 0\le R_A\le R_E\\ \left(-\dfrac{1}{R_E}\right)^2+\left(\dfrac{1}{R_E}\right)^2 & \text{if } R_E\le R_A\le 2R_E\end{cases}$$
$$\Leftrightarrow\left|\frac{dZ}{dR_A}\right|^2=\frac{1}{R_E^2}+\frac{1}{R_E^2}=\frac{2}{R_E^2}\quad \text{for } 0\le\forall R_A\le 2R_E$$

$\Leftrightarrow\left|\dfrac{dZ}{dR_A}\right|=\dfrac{\sqrt{2}}{R_E}=\text{constant}>0$ if $R_E>0$; that means that the module of the slope of the complex probability vector *Z* in $\mathcal{C}$ or of its rate of change is constant and positive and it depends only on R_E ; hence, we have a constant increase in $\mathrm{Re}(Z)$ and a constant decrease in $\mathrm{Im}(Z)$ as functions of the iterations number *N* and as *Z* goes from (0, *i*) at *N* = 0 till (1,0) at the simulation end; hence, till $\mathrm{Re}(Z)=P_r$ reaches the value 1 that means till the random experiment converges to R_E.

Furthermore, since $Pc^2=DOK-Chf=DOK+MChf=1$ then $Pc=1=$ Probability of convergence in $\mathcal{C}$ and consequently : $\dfrac{d(Pc)}{dR_A}=\dfrac{d(1)}{dR_A}=0$, that means that *Pc* is constantly equal to 1 for every value of R_A, of R_E, and of the iterations number *N*, that means for any stochastic

experiment and for any simulation of Monte Carlo method. So, we conclude that in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ we have complete and perfect knowledge of the random experiment which has become now a deterministic one since the extension in the complex probability plane $\mathcal{C}$ defined by the *CPP* axioms has changed all stochastic variables to deterministic variables.

V- The Evaluation of the New Paradigm Parameters

We can deduce from what has been elaborated previously the following:

The real convergence probability: $P_r(N) = 1 - \left| \frac{R_E - R_A(N)}{R_E} \right|$

We have $0 \le N \le N_C$ where $N = 0$ corresponds to the instant before the beginning of the random experiment when $R_A(N=0) = 0$ or $= 2R_E$, and where $N = N_C$ (iterations number needed for the method convergence) corresponds to the instant at the end of the random experiments and Monte Carlo methods when $R_A(N = N_C) \to R_E$.

The imaginary complementary divergence probability: $P_m(N) = i \left| \frac{R_E - R_A(N)}{R_E} \right|$

The real complementary divergence probability: $P_m(N)/i = \left| \frac{R_E - R_A(N)}{R_E} \right|$

The complex probability and random vector:

$$Z(N) = P_r(N) + P_m(N) = \left[1 - \left| \frac{R_E - R_A(N)}{R_E} \right| \right] + i \left| \frac{R_E - R_A(N)}{R_E} \right|$$

The Degree of Our Knowledge:

$$DOK(N) = |Z(N)|^2 = P_r^2(N) + [P_m(N)/i]^2 = \left[1 - \left| \frac{R_E - R_A(N)}{R_E} \right| \right]^2 + \left[\left| \frac{R_E - R_A(N)}{R_E} \right| \right]^2$$

$$= 1 + 2iP_r(N)P_m(N) = 1 - 2P_r(N)[1 - P_r(N)] = 1 - 2P_r(N) + 2P_r^2(N)$$

$$= 1 - 2\left| \frac{R_E - R_A(N)}{R_E} \right| + 2\left[\frac{R_E - R_A(N)}{R_E} \right]^2$$

$DOK(N)$ is equal to 1 when $P_r(N) = P_r(0) = 0$ and when $P_r(N) = P_r(N_C) = 1$.

The Chaotic Factor:

$$Chf(N) = 2iP_r(N)P_m(N) = -2P_r(N)[1 - P_r(N)] = -2P_r(N) + 2P_r^2(N)$$

$$= -2\left| \frac{R_E - R_A(N)}{R_E} \right| + 2\left[\frac{R_E - R_A(N)}{R_E} \right]^2$$

$Chf(N)$ is null when $P_r(N) = P_r(0) = 0$ and when $P_r(N) = P_r(N_C) = 1$.

The Magnitude of the Chaotic Factor *MChf*:

$$MChf(N) = |Chf(N)| = -2iP_r(N)P_m(N) = 2P_r(N)[1 - P_r(N)] = 2P_r(N) - 2P_r^2(N)$$
$$= 2\left|\frac{R_E - R_A(N)}{R_E}\right| - 2\left[\frac{R_E - R_A(N)}{R_E}\right]^2$$

$MChf(N)$ is null when $P_r(N) = P_r(0) = 0$ and when $P_r(N) = P_r(N_C) = 1$.

At any iteration number *N*: $0 \le \forall N \le N_C$, the probability expressed in the complex probability set $\mathcal{C}$ is the following:

$$Pc^2(N) = [P_r(N) + P_m(N)/i]^2 = |Z(N)|^2 - 2iP_r(N)P_m(N)$$
$$= DOK(N) - Chf(N)$$
$$= DOK(N) + MChf(N)$$
$$= 1$$

then,

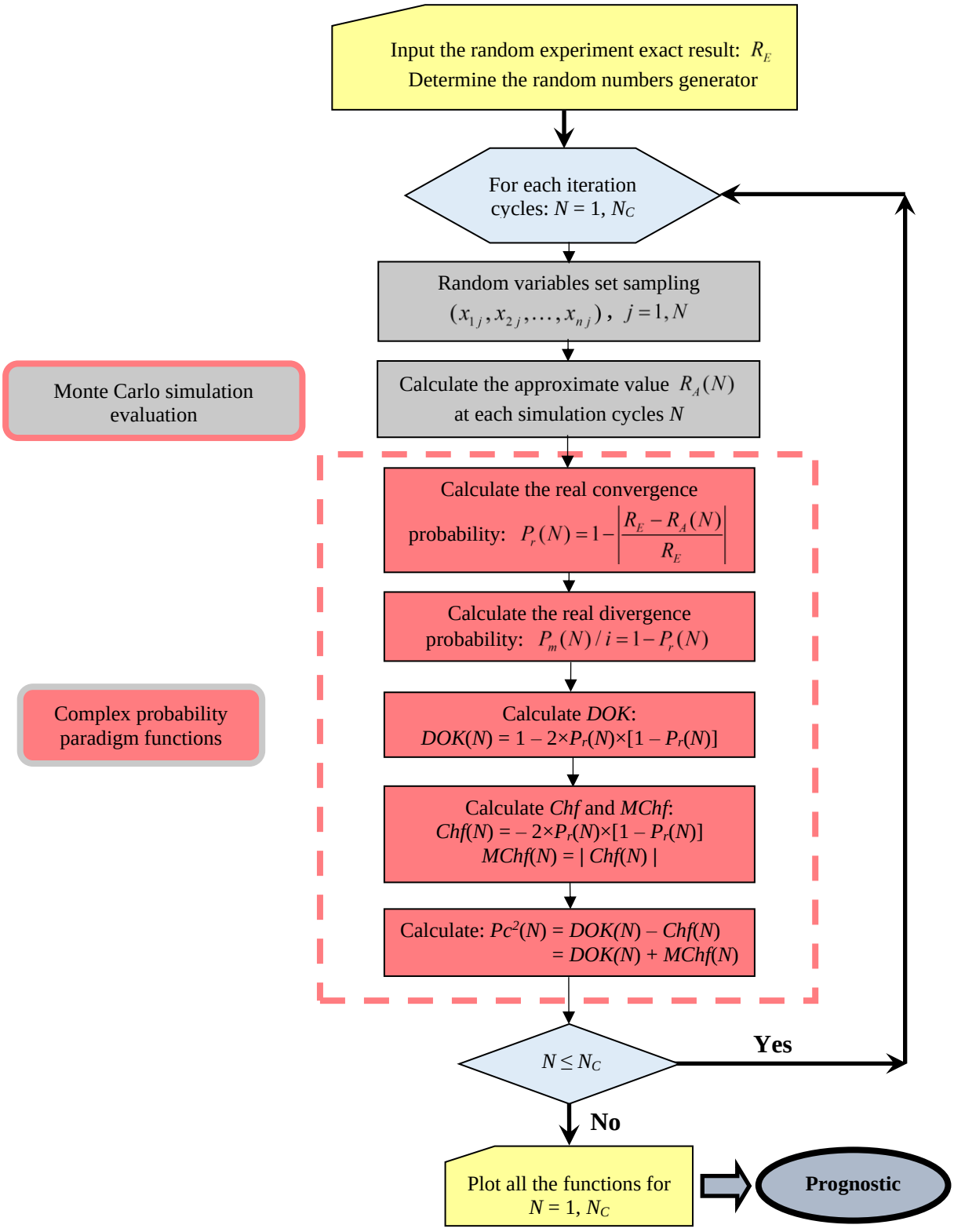
$$Pc^2(N) = [P_r(N) + P_m(N)/i]^2 = \{P_r(N) + [1 - P_r(N)]\}^2 = 1^2 = 1 \Leftrightarrow Pc(N) = 1 \text{ always}$$

Hence, the prediction of the convergence probabilities of the stochastic Monte Carlo experiments in the set $\mathcal{C}$ is permanently certain.

Let us consider thereafter a multidimensional integral and a stochastic experiment to simulate the Monte Carlo methods and to draw, to visualize, as well as to quantify all the *CPP* and prognostic parameters.

VI- Flowchart of the Complex Probability and Monte Carlo Techniques Prognostic Model

The following flowchart summarizes all the procedures of the proposed complex probability prognostic model:



VII- Simulation of the New Paradigm

Note that all the numerical values found in the simulations of the new paradigm for any iteration cycles *N* were computed using the 64-Bit MATLAB version 2024 software and compared to the values found by Microsoft Visual C++ programs. In addition, the reader should take care of the rounding and truncation errors since all numerical values are represented by at most five significant digits and since we are using Monte Carlo methods of integration and simulation which give approximate results subject to random effects and fluctuations. We have considered for this purpose a high-capacity computer system: a workstation computer with parallel microprocessors, a 64-Bit operating system, and a 64-GB RAM.

VII-1- The Continuous Random Case: A Four-Dimensional Multiple Integral

The Monte Carlo technique of integration can be summarized by the following equation:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\dots\int_{a_n}^{b_n} f(x_1,x_2,\dots,x_n).dx_1dx_2\dots dx_n \cong \frac{[(b_1-a_1)\times(b_2-a_2)\times\dots\times(b_n-a_n)]}{N}\sum_{j=1}^{N} f(x_{1j},x_{2j},\dots,x_{nj})$$

Let us consider here the multidimensional integral of the following function:

$$\int_0^{4/3}\int_0^{4/3}\int_0^{4/3}\int_0^{4/3} xyzw.dxdydzdw = \int_0^{4/3}\int_0^{4/3}\int_0^{4/3}\left[\frac{x^2}{2}\right]_0^{4/3} yzw.dydzdw = \int_0^{4/3}\int_0^{4/3}\int_0^{4/3}\frac{16}{18}yzw.dydzdw$$

$$= \frac{8}{9}\int_0^{4/3}\int_0^{4/3}\left[\frac{y^2}{2}\right]_0^{4/3} zw.dzdw = \frac{8}{9}\int_0^{4/3}\int_0^{4/3}\frac{16}{18}zw.dzdw = \frac{64}{81}\int_0^{4/3}\left[\frac{z^2}{2}\right]_0^{4/3} w.dw = \frac{64}{81}\int_0^{4/3}\frac{16}{18}w.dw$$

$$= \frac{512}{729}\left[\frac{w^2}{2}\right]_0^{4/3} = \frac{512}{729}\times\frac{16}{18} = \frac{512}{729}\times\frac{8}{9} = \frac{4,096}{6,561} = 0.62429507696997411\dots$$

$\Leftrightarrow R_E = 0.62429507696997411\dots$ by the deterministic methods of calculus.

$\Leftrightarrow f(x,y,z,w) = xyzw$, where *x*, *y*, *z*, and *w* follow a discrete uniform distribution *U* such that:

$$x \mapsto U(0,4/3),\ y \mapsto U(0,4/3),\ z \mapsto U(0,4/3),\ w \mapsto U(0,4/3)$$

$$\Leftrightarrow \int_0^{4/3}\int_0^{4/3}\int_0^{4/3}\int_0^{4/3} xyzw.dxdydzdw \cong \frac{[(4/3-0)\times(4/3-0)\times(4/3-0)\times(4/3-0)]}{N}\sum_{j=1}^{N} x_j y_j z_j w_j$$

$$= \frac{256/81}{N}\sum_{j=1}^{N} x_j y_j z_j w_j = R_A$$

with $1 \le N \le N_C$ after applying Monte Carlo method.

Moreover, the four figures (Figures 3-6) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E = 0.62429507696997411\dots$ for *N* = 50, 100, 500, and $N = N_C = 100,000$ iterations. Therefore, we have:

$\lim_{N\to+\infty} P_r(N) = \lim_{N\to+\infty}\left\{1-\left|\frac{R_E - R_A(N)}{R_E}\right|\right\} = 1-\left|\frac{R_E - R_E}{R_E}\right| = 1-0 = 1$ which is equal to the convergence probability of Monte Carlo method as $N \to +\infty$.

Additionally, Figure 7 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters ($Chf, R_A, P_r, MChf, R_E, DOK, P_m/i, Pc$) after applying it to this four-dimensional integral.

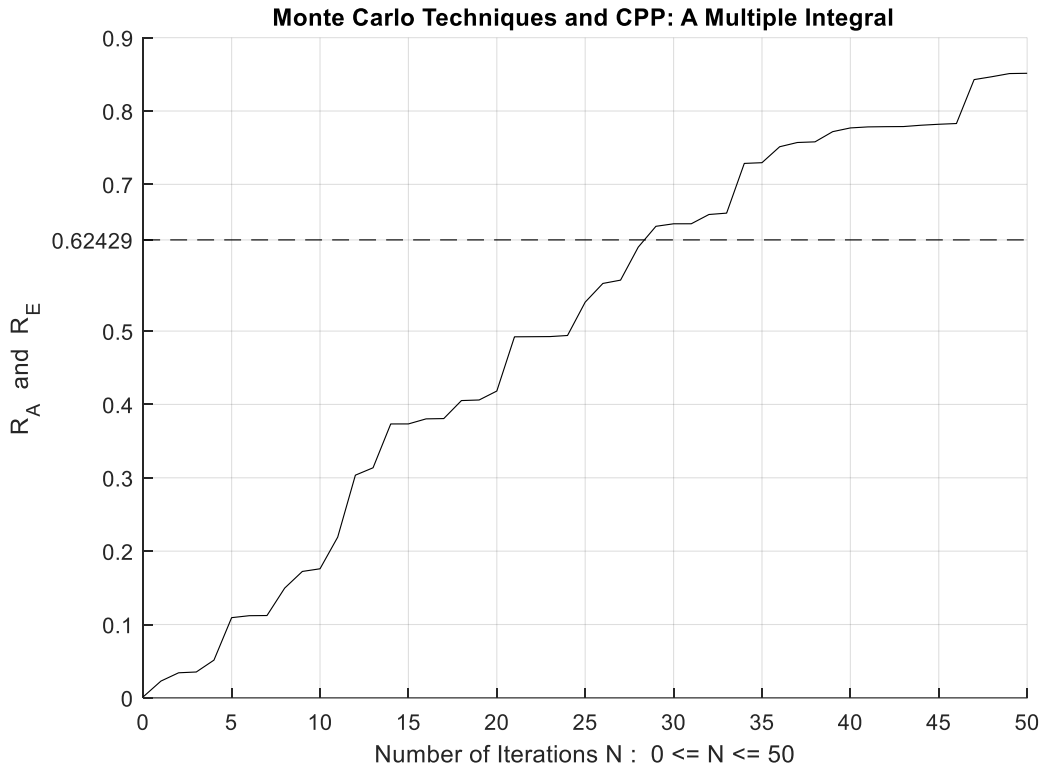


Figure 3. The increasing convergence of the Monte Carlo method up to $N = 50$ iterations.

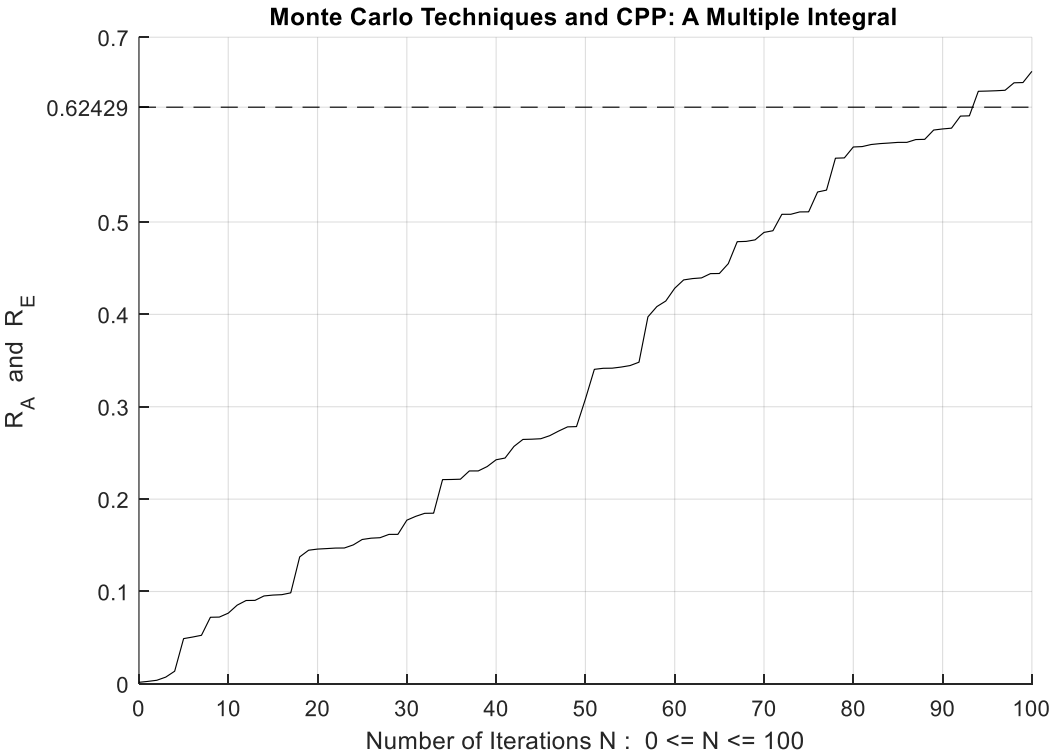


Figure 4. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

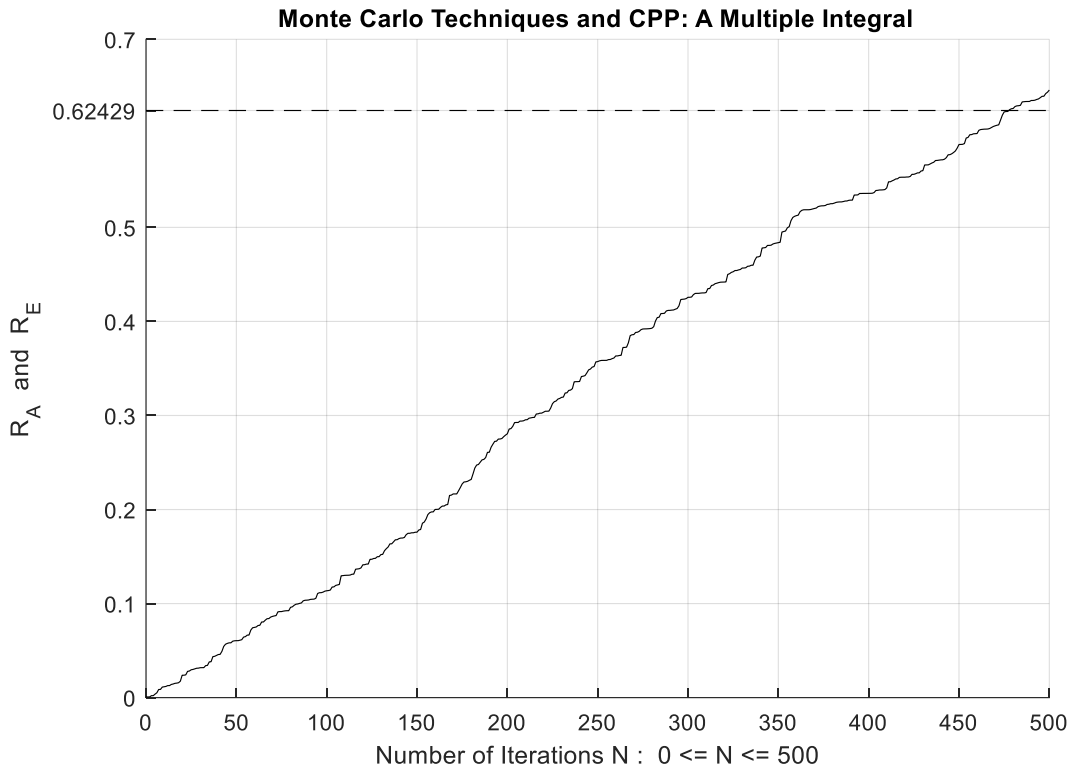


Figure 5. The increasing convergence of the Monte Carlo method up to N = 500 iterations.

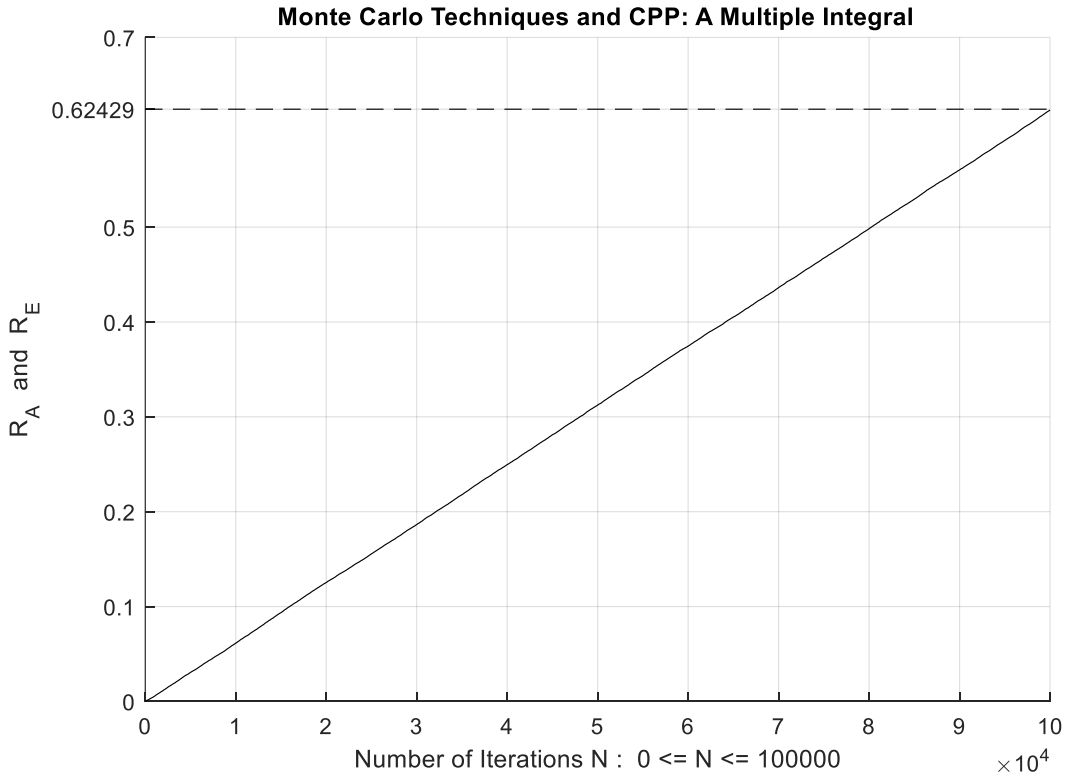


Figure 6. The increasing convergence of the Monte Carlo method up to N = 100,000 iterations.

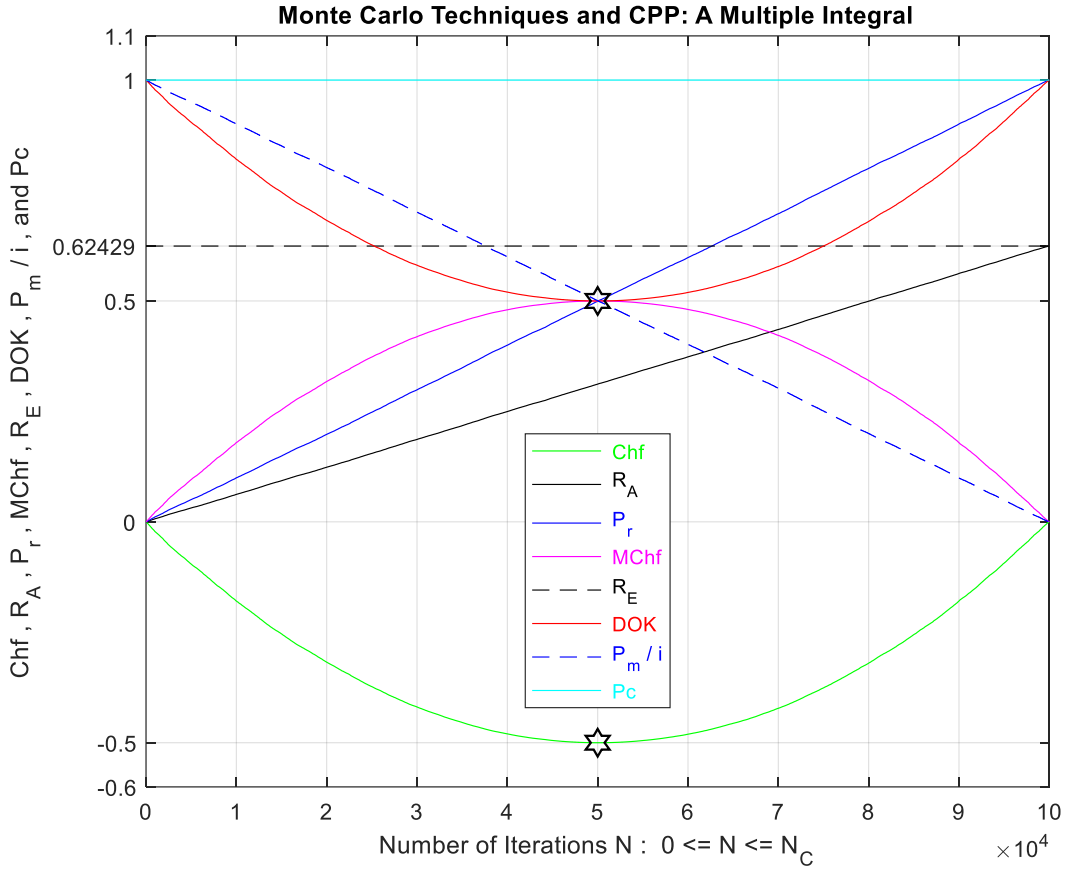


Figure 7. The *CPP* parameters and the Monte Carlo method for a multiple integral.

VII-2- The Discrete Random Case: The Matching Birthday Problem

An interesting problem that can be solved using simulation is *the famous birthday problem.* Suppose that in a room of *n* persons, each of the 365 days of the year (not a leap year) is equally likely to be someone's birthday. From probability theory, it can be shown that, contrary to intuition, only 23 persons need to be present for the chances to be better than fifty-fifty that at least two of them will have the same birthday!

Many people are curious about the theoretical reasoning behind this result, so we discuss it briefly before solving the simulation problem. After someone is asked his or her birthday, the chances that the next person asked will not have the same birthday are 364/365. The chances that the third person's birthday will not match those of the first two people are 363/365. The chances of two successive independent events occurring is the product of the probability of the separate events. In general, the probability that the n^{th} person asked will have a birthday different from that of anyone already asked is:

$$P(\text{all } n \text{ birthdays are different}) = \left(\frac{365}{365}\right)\times\left(\frac{364}{365}\right)\times\left(\frac{363}{365}\right)\times\ldots\times\left(\frac{365-(n-1)}{365}\right)$$

The probability that the n^{th} person asked will provide a match is 1 minus this value:

$$P(\text{matching birthdays}) =$$

$$1-\left(\frac{365}{365}\right)\times\left(\frac{364}{365}\right)\times\left(\frac{363}{365}\right)\times\ldots\times\left(\frac{365-(n-1)}{365}\right)=1-\frac{(365)\times(364)\times(363)\times\ldots\times[365-(n-1)]}{365^n}=R_E$$

shows that with 23 persons the chances are 50.7%; with 55 persons, the chances are 98.6% or almost theoretically certain that at least two out of 55 people will have the same birthday. The table below gives the theoretical probabilities of matching birthdays for a selected number of people *n* (Table 1).

Number of People *n*	Theoretical Probability = R_E
***n* = 5**	***P* = 0.027135573700**
***n* = 10**	***P* = 0.116948177711**
***n* = 15**	***P* = 0.252901319764**
***n* = 20**	***P* = 0.411438383581**
***n* = 22**	***P* = 0.475695307663**
***n* = 23**	***P* = 0.507297234324**
***n* = 25**	***P* = 0.568699703969**
***n* = 30**	***P* = 0.706316242719**
***n* = 35**	***P* = 0.814383238875**
***n* = 40**	***P* = 0.891231809818**
***n* = 45**	***P* = 0.940975899466**
***n* = 50**	***P* = 0.970373579578**
***n* = 55**	***P* = 0.986262288816**
***n* = 100**	***P* = 0.999999692751**
***n* = 133**	***P* = 0.999999999999**
***n* = 365**	***P* = 1.000000000000**

Table 1. Some theoretical probabilities of matching birthdays for *n* people where $1\le n\le 365$.

Without using probability theory, we can write a routine that uses the random-number generator to compute the approximate chances for groups of *n* persons. Clearly, all that is needed is to select *n* random integers from the set {1, 2, 3, …, 365} and to examine them in some way to determine whether there is a match. By repeating this experiment a large number of times, we can compute the probability of at least one match in any gathering of *n* persons. Note that if $n\ge 366$ then $P(\text{matching birthdays})=1$ by the famous pigeonhole principle. Moreover, the four figures (Figures 8-11) show the increasing convergence of Monte Carlo method and simulation to the exact result $R_E=0.706316242719...$ for $n=30$ people and for N = 50, 100, 500, and $N=N_C=750,000$ iterations. Therefore, we have:

$\lim_{N\to+\infty} P_r(N)=\lim_{N\to+\infty}\left\{1-\left|\frac{R_E-R_A(N)}{R_E}\right|\right\}=1-\left|\frac{R_E-R_E}{R_E}\right|=1-0=1$ which is equal to the convergence probability of Monte Carlo method as $N\to+\infty$.

Additionally, Figure 12 illustrates clearly and visibly the relation of Monte Carlo method to the complex probability paradigm with all its parameters $(Chf, R_A, P_r, MChf, R_E, DOK, P_m/i, Pc)$ after applying it to this matching birthday problem.

The algorithm in Microsoft Visual C++ is the following:

```
// The birthday problem.
#include <iostream>
#include <iomanip>
#include <cstdlib>
#include <ctime>
#include <cmath>

using namespace std;

void theoretical(long int);
void simulated(long int);

int main()
{
      long int n;

      cout << "                              THE BIRTHDAY PROBLEM" << endl;
      cout << "                             ----------------------\n"
           << endl;
      cout << "Input the number of persons : ";
      cin >> n;
      cout << endl;

      cout << fixed << setprecision(5);

      theoretical(n);
      simulated(n);

      return 0;
}

void theoretical(long int n)
{
      long int i;
      long double prod;
      long double P;

      prod = 1;
      for (i = 0; i <= (n - 1); i++)
            prod = (long double) prod * (365 - i) / 365;

      P = 1 - prod;
```

```
	cout << "The theoretical probability of a match for n = " << n
	     << " is = " << P << endl;
}

void simulated(long int n)
{
	long int table[366] = { 0 };
	long int random, match;
	long double P, sum, limit;
	long int i, j;

	srand(time(0));
	limit = 100000000;
	sum = 0; random = 0; P = 0; match = 0;

	for (j = 1; j <= limit; j++)
	{
		for (i = 1; i <= n; i++)
		{
			random = 1 + rand() % 365;
			table[random] += 1;
			random = 0;
		}

		i = 0;
		while ((i <= 365) && (match == 0))
		{
			if (table[i] >= 2)
				match = 1;
			++i;
		}

		if (match == 1)
			sum += 1;

		for (i = 0; i <= 365; i++)
			table[i] = 0;

		match = 0;
	}

	P = (long double) sum / limit;
	cout << "The simulated   probability of a match for n = " << n
	     << " is = " << P
	     << "\n" << endl;
}
```

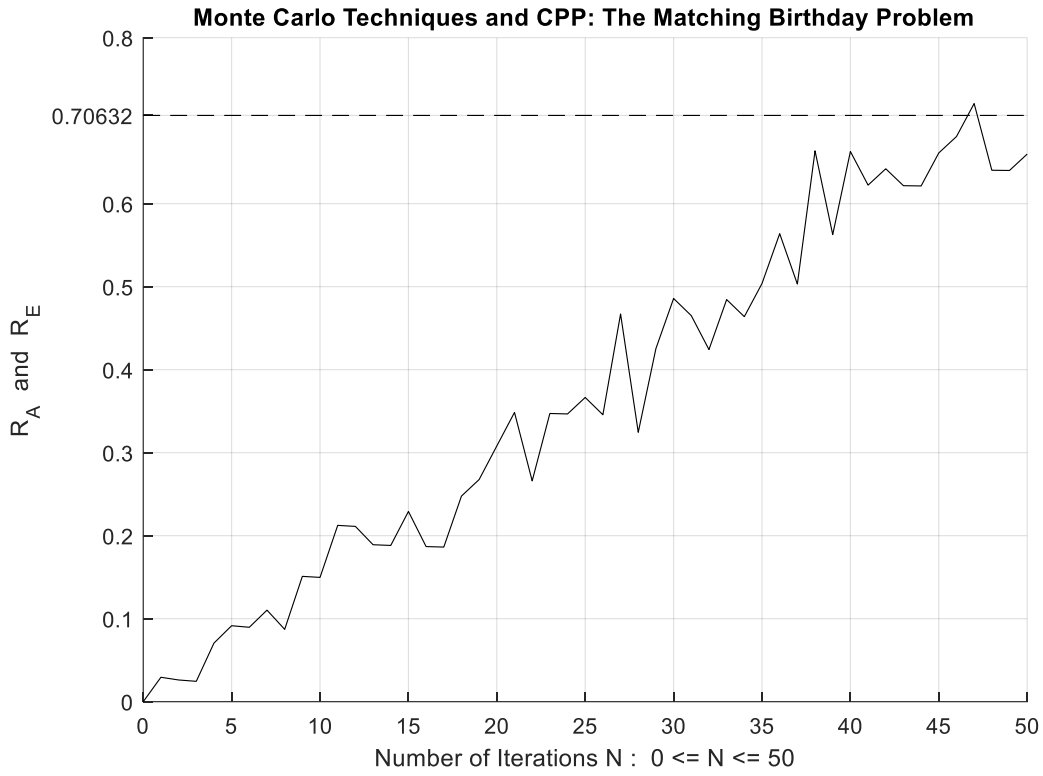


Figure 8. The increasing convergence of the Monte Carlo method up to $N = 50$ iterations.

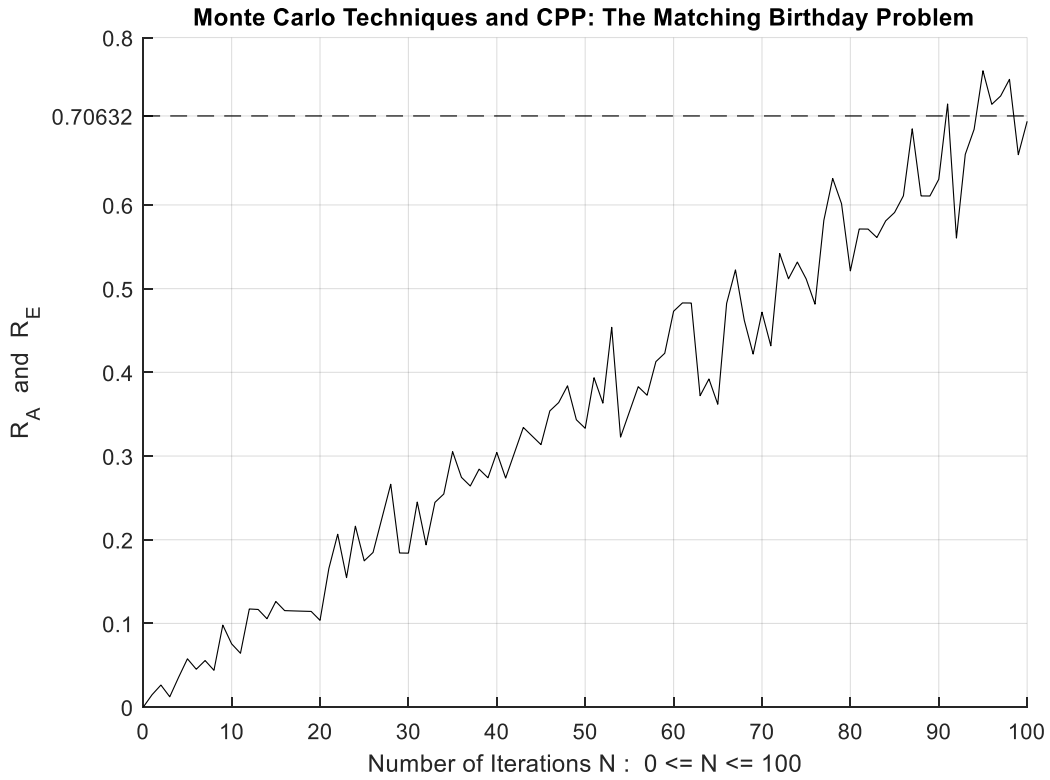


Figure 9. The increasing convergence of the Monte Carlo method up to $N = 100$ iterations.

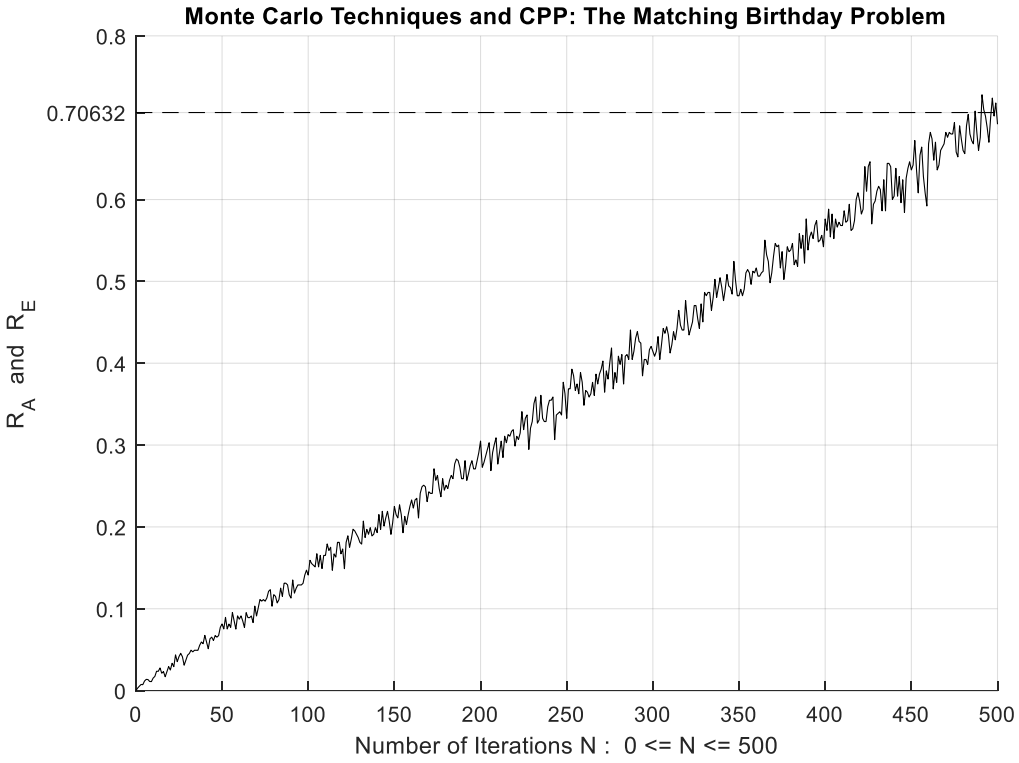


Figure 10. The increasing convergence of the Monte Carlo method up to N = 500 iterations.

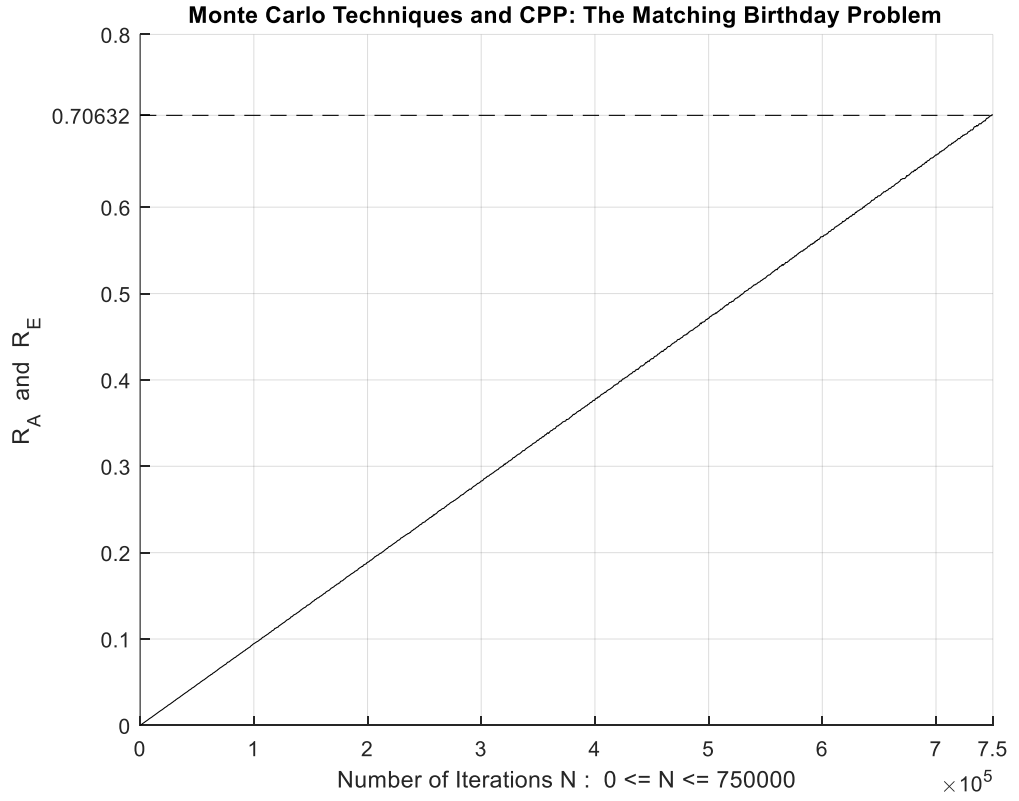


Figure 11. The increasing convergence of the Monte Carlo method up to N = 750,000 iterations.

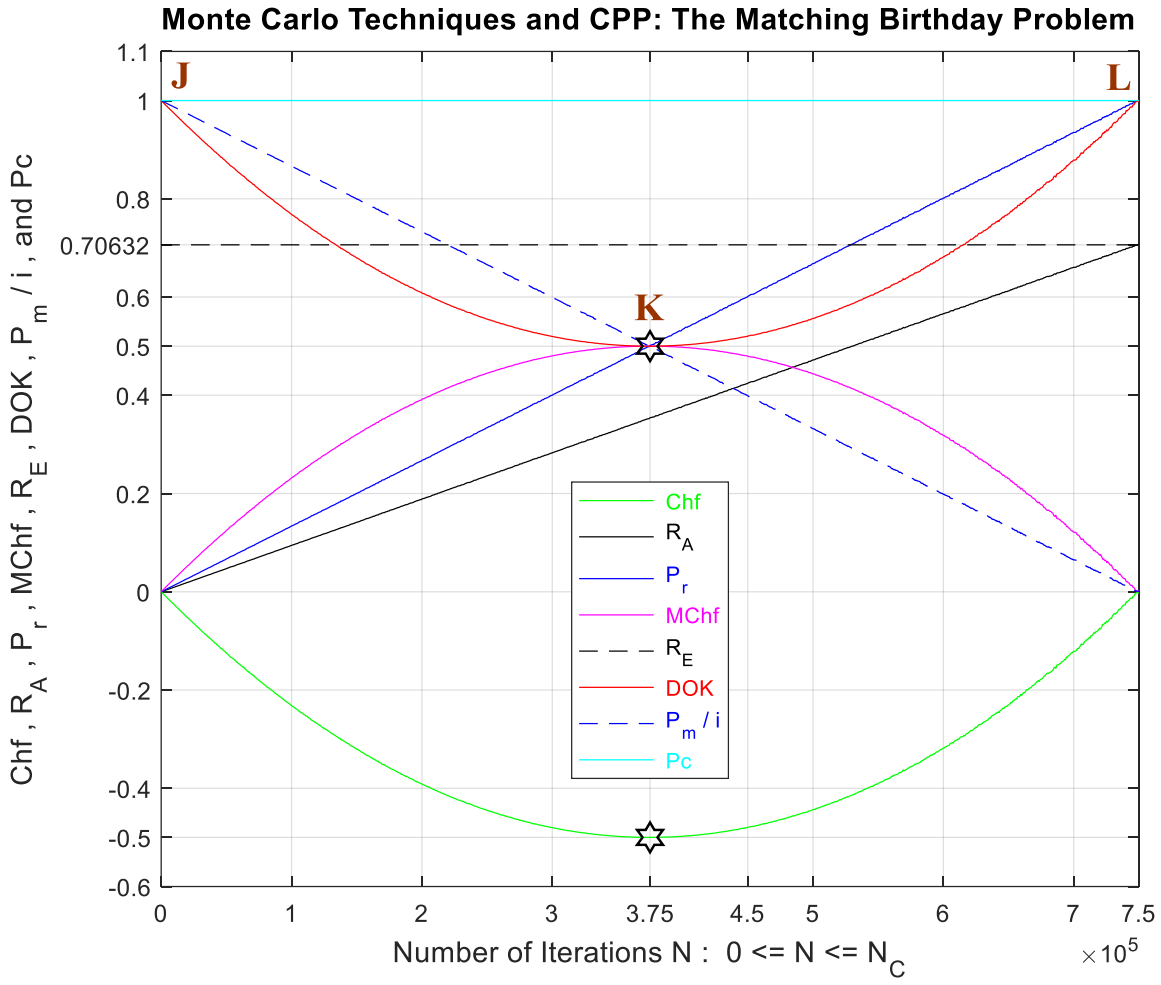


Figure 12. The *CPP* parameters and the Monte Carlo techniques for the matching birthday problem.

VII-2-1 The Complex Probability Cubes

In the first cube (Figure 13), the simulation of *DOK* and *Chf* as functions of each other and of the iterations *N* for the matching birthday problem can be seen. The line in cyan is the projection of $Pc^2(N) = DOK(N) - Chf(N) = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point J ($DOK = 1$, $Chf = 0$) when $N = 0$ iterations, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when $N = 375{,}000$ iterations, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $N = N_C = 750{,}000$ iterations. The other curves are the graphs of $DOK(N)$ (red) and $Chf(N)$ (green, blue, pink) in different planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, $N = 375{,}000$ iterations). The point L corresponds to ($DOK = 1$, $Chf = 0$, $N = N_C = 750{,}000$ iterations). The three points J, K, L are the same as in Figure 12.

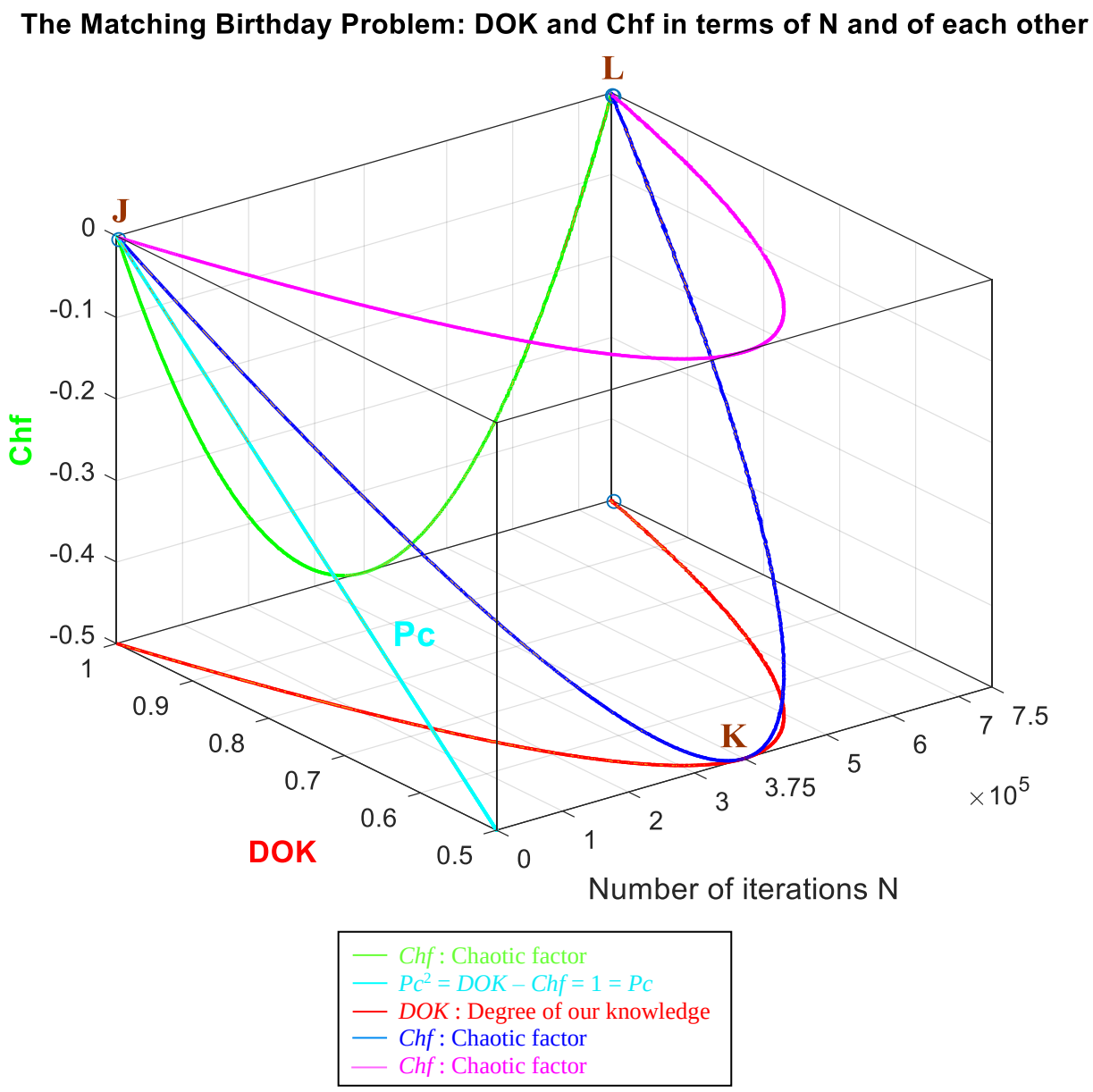


Figure 13. *DOK* and *Chf* in terms of *N* and of each other for the matching birthday problem.

In the second cube (Figure 14), we can notice the simulation of the convergence probability $P_r(N)$ and its complementary real divergence probability $P_m(N)/i$ in terms of the iterations N for the matching birthday problem. The line in cyan is the projection of $Pc^2(N) = P_r(N) + P_m(N)/i = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point ($P_r = 0$, $P_m/i = 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $P_r(N)$ in the plane $P_r(N) = P_m(N)/i$. This curve starts at the point J ($P_r = 0$, $P_m/i = 1$, $N = 0$ iterations), reaches the point K ($P_r = 0.5$, $P_m/i = 0.5$, $N = 375{,}000$ iterations), and gets at the end to L ($P_r = 1$, $P_m/i = 0$, $N = N_C = 750{,}000$ iterations). The blue curve represents $P_m(N)/i$ in the plane $P_r(N) + P_m(N)/i = 1$. Notice the importance of the point K which is the intersection of the red and blue curves at $N = 375{,}000$ iterations and when $P_r(N) = P_m(N)/i = 0.5$. The three points J, K, L are the same as in Figure 12.

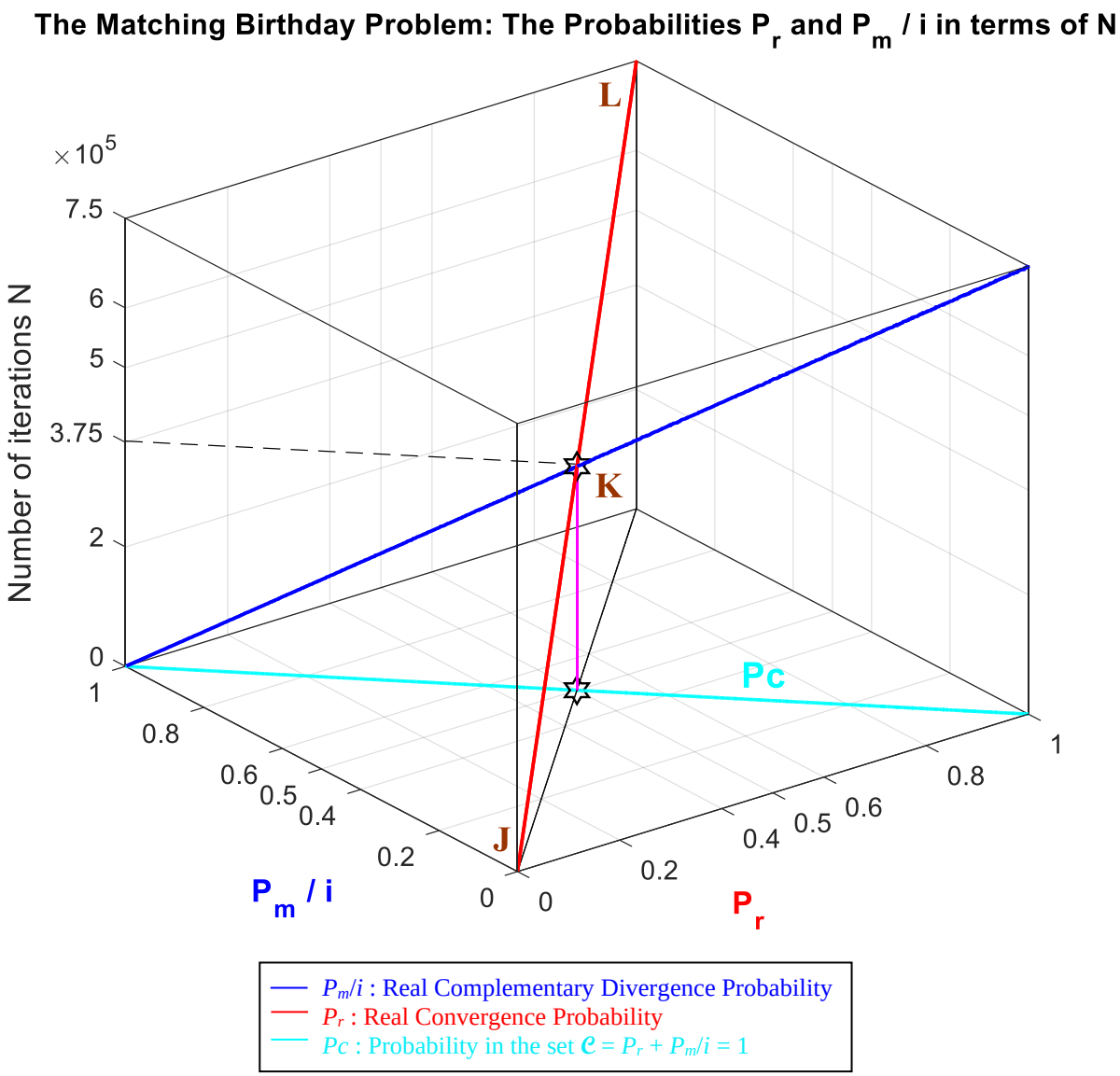


Figure 14. P_r and P_m/i in terms of N and of each other for the matching birthday problem.

In the third cube (Figure 15), we can notice the simulation of the complex random vector $Z(N)$ in $\mathcal{C}$ as a function of the real convergence probability $P_r(N) = \text{Re}(Z)$ in $\mathcal{R}$ and of its complementary imaginary divergence probability $P_m(N) = i\times\text{Im}(Z)$ in $\mathcal{M}$, and this in terms of the iterations N for the matching birthday problem. The red curve represents $P_r(N)$ in the plane $P_m(N) = 0$ and the blue curve represents $P_m(N)$ in the plane $P_r(N) = 0$. The green curve represents the complex probability vector $Z(N) = P_r(N) + P_m(N) = \text{Re}(Z) + i\times\text{Im}(Z)$ in the plane $P_r(N) = iP_m(N) + 1$. The curve of $Z(N)$ starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point L ($P_r = 1$, $P_m = 0$, $N = N_C = 750{,}000$ iterations). The line in cyan is $P_r(0) = iP_m(0) + 1$ and it is the projection of the $Z(N)$ curve on the complex probability plane whose equation is $N = 0$ iterations. This projected line starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point ($P_r = 1$, $P_m = 0$, $N = 0$ iterations). Notice the importance of the point K corresponding to $N = 375{,}000$ iterations and when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in Figure 12.

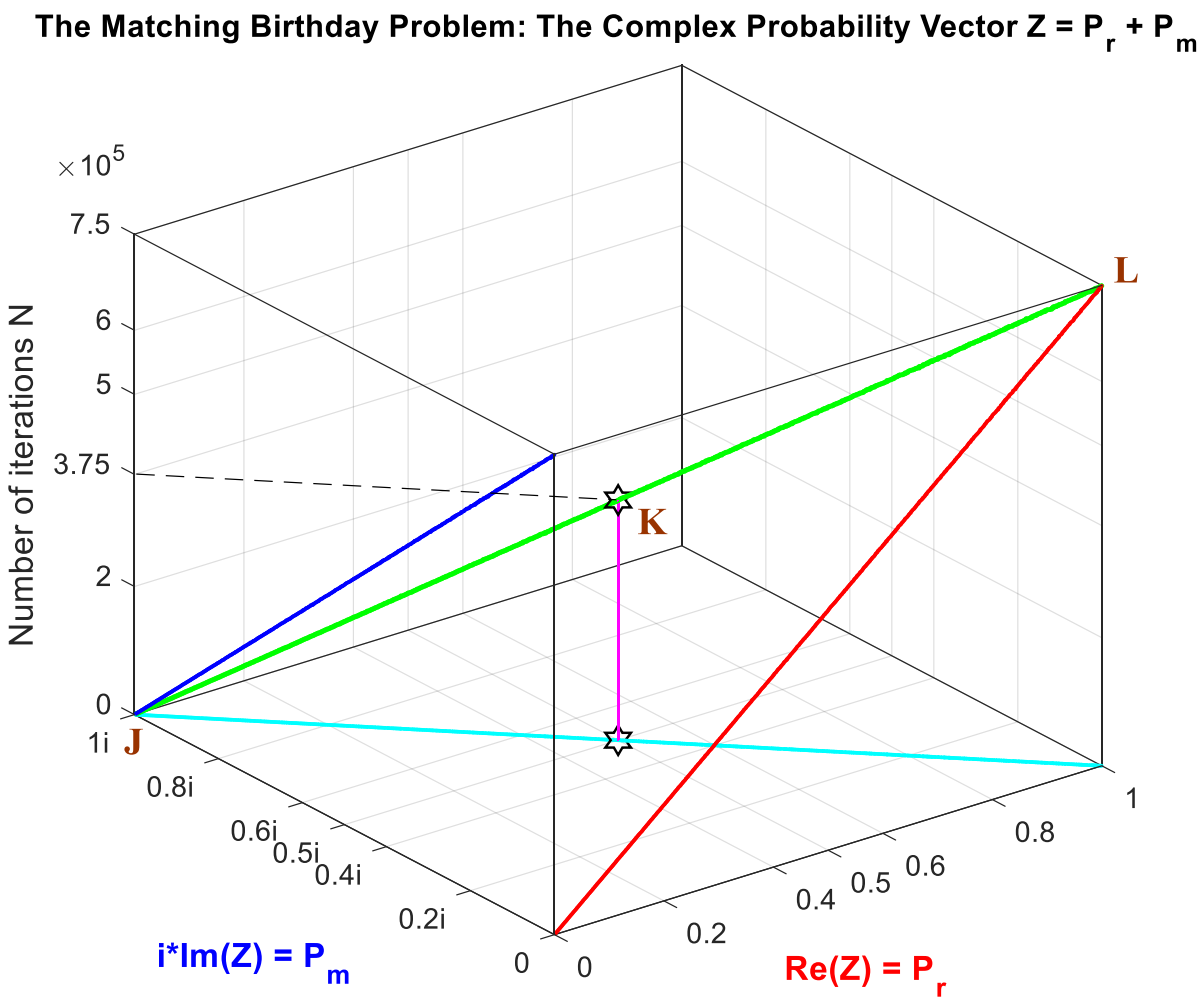


Figure 15. The Complex Probability Vector Z in terms of N for the matching birthday problem.

VIII- Conclusion and Perspectives

In the current research chapter, the original extended Kolmogorov model of eight axioms (*EKA*) was connected and applied to the classical and random Monte Carlo numerical techniques. Thus, a tight link between Monte Carlo algorithms and the novel paradigm was executed. Accordingly, the model of "Complex Probability" was more expanded beyond the scope of my earlier research studies on this subject.

Also, as it was verified and demonstrated in the original model, when $N = 0$ (before the random simulation beginning) and when $N = N_C$ (when Monte Carlo algorithm converges to the exact result) then the degree of our knowledge (*DOK*) is 1 and the chaotic factor (*Chf* and *MChf*) is 0

since the stochastic effects and fluctuations have either not commenced yet or they have terminated their task on the random experiment. During the course of the nondeterministic experiment ($N > 0$) we have: $0.5 \leq DOK < 1$, $-0.5 \leq Chf < 0$, and $0 < MChf \leq 0.5$. We notice that during this entire process we have continually and incessantly $Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$, that means that the simulation which looked to be random and nondeterministic in the set $\mathcal{R}$ is now deterministic and certain in the set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after adding the contributions of $\mathcal{M}$ to the experiment happening in $\mathcal{R}$ and thus after removing and subtracting the chaotic factor from the degree of our knowledge. Additionally, the probabilities of convergence and divergence of the random Monte Carlo procedure that correspond to each iteration cycle N have been determined in the three sets of probabilities which are $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$ by P_r, P_m, and Pc respectively. Subsequently, at each instance of N, the novel Monte Carlo techniques and *CPP* parameters R_E, R_A, P_r, P_m, P_m / i, DOK, Chf, $MChf$, Pc, and Z are perfectly and surely predicted in the set of complex probabilities $\mathcal{C}$ with Pc kept as equal to 1 continuously and forever.

Furthermore, using all these shown simulations and obtained graphs all over the entire research chapter, we can visualize and quantify both the certain knowledge (expressed by *DOK* and *Pc*) and the system chaos and stochastic influences and effects (expressed by *Chf* and *MChf*) of Monte Carlo algorithms. This is definitely very wonderful, fruitful, and fascinating and demonstrates once again the advantages of extending the five axioms of probability of Kolmogorov and thus the benefits and novelty of this original theory in applied mathematics and prognostics that can be called verily:

"The Complex Probability Paradigm".

Moreover, it is important to state here that one essential and very well-known probability distribution was taken into consideration in the current chapter which is the uniform and discrete probability distribution as well as a specific generator of uniform random numbers, knowing that the original *CPP* model can be applied to any generator of uniform random numbers that exists in literature. This will yield certainly to analogous results and conclusions and will confirm without any doubt the success of my innovative theory.

As a prospective and future challenges and research, we intend to more develop the novel conceived prognostic paradigm and to apply it to a diverse set of nondeterministic events like for other stochastic phenomena as in the classical theory of probability and in stochastic processes. Additionally, we will implement *CPP* also to other stochastic problems which have huge consequences when applied to economics, to chemistry, to physics, to pure and applied mathematics.

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this research work.

CHAPTER THREE

THE PARADIGM OF COMPLEX PROBABILITY AND LE COMTE DE BUFFON'S NEEDLE

"Nature is an infinite sphere of which the center is everywhere and the circumference nowhere."

Blaise Pascal.

"I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our "creations," are simply the notes of our observations."

Godfrey Harold Hardy.

"*Imagination decides everything.*"

Blaise Pascal.

"The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics."

Godfrey Harold Hardy.

Abstract: In the current work, we extend and incorporate in the five-axioms probability system of *Andrey Nikolaevich Kolmogorov* set up in 1933 the imaginary set of numbers and this by adding three supplementary axioms. Consequently, any stochastic experiment can thus be achieved in the extended complex probabilities set $\mathcal{C}$ which is the sum of the real probabilities set $\mathcal{R}$ and the imaginary probabilities set $\mathcal{M}$. The purpose here is to evaluate the complex probabilities by considering additional novel imaginary dimensions to the experiment occurring in the "real" laboratory. Therefore, the random phenomenon outcome and result in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ can be predicted absolutely and perfectly no matter what the random distribution of the input variable in $\mathcal{R}$ is since the associated probability in the entire set $\mathcal{C}$ is constantly and permanently equal to one. Thus, the following consequence indicates that chance and randomness in $\mathcal{R}$ is replaced now by absolute and total determinism in $\mathcal{C}$ as a result of subtracting from the degree of our knowledge the chaotic factor in the probabilistic experiment. Moreover, we will apply this innovative paradigm to the well-known Buffon's needle technique and to its random algorithms and procedures in a novel way.

Keywords: Degree of our knowledge, Chaotic factor, Magnitude of chaotic factor, Complex random vector, Probability norm, Real and Imaginary Probabilities, Complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, Convergence probability, Divergence probability, Simulation.

NOMENCLATURE

$\mathcal{R}$ = the events real set
$\mathcal{M}$ = the events imaginary set
$\mathcal{C}$ = the events complex set
i = the imaginary number with $i^2 = -1$ or $i = \sqrt{-1}$
EKA = Extended Kolmogorov's Axioms
CPP = Complex Probability Paradigm
P_{rob} = any event probability
P_r = the probability in the real set $\mathcal{R}$ = convergence probability in $\mathcal{R}$
P_m = the probability in the complementary imaginary set $\mathcal{M}$ that corresponds to the real probability set in $\mathcal{R}$ = divergence probability in $\mathcal{M}$
Pc = the event probability in $\mathcal{R}$ with its associated event in $\mathcal{M}$ = probability in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$
EX = the random experiment exact result
AP = the random experiment approximate result
Z = complex probability number = complex random vector = sum of P_r and P_m
DOK = $|Z|^2$ = the degree of our knowledge of the stochastic experiment or system, it is the square of the norm of *Z*
Chf = the chaotic factor of *Z*
MChf = the magnitude of the chaotic factor of *Z*
N = the number of iterations cycles = number of random vectors
N_C = the number of iterations cycles till the convergence of Buffon's needle method to *EX* = the number of random vectors till convergence.

1- Introduction [1-90]

All our work in classical probability theory is to compute probabilities. The original idea in this book is to add new dimensions to our random experiment which will make the work totally deterministic. In fact, probability theory is a nondeterministic theory by nature that means that the outcome of the stochastic events is due to chance and luck. By adding new dimensions to the event occurring in the "real" laboratory which is $\mathcal{R}$, we make the work deterministic and hence a random experiment will have a certain outcome in the complex set of probabilities $\mathcal{C}$. It is of great importance that stochastic systems become totally predictable since we will be perfectly knowledgeable to foretell the outcome of all chaotic and random events that occur in nature like for example in statistical mechanics, in all stochastic processes, or in the well-established field of prognostic. Therefore, the work that should be done is to add to the real set of probabilities $\mathcal{R}$, the contributions of $\mathcal{M}$ which is the imaginary set of probabilities which will make the event in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ absolutely deterministic. If this is found to be fruitful, then a new theory in stochastic sciences and prognostic would be elaborated and this to understand deterministically those phenomena that used to be random phenomena in $\mathcal{R}$. This is what I called "The Complex Probability Paradigm (*CPP*)" that was initiated and elaborated in my previous papers and works.

2- The Purpose and the Advantages of the Current Chapter [37-90]

The advantages and the purpose of the present chapter are to:

1- Extend the theory of classical probability to cover the complex numbers set, hence to connect the probability theory to the field of complex variables and analysis. This task was started and elaborated in my earlier papers and works.

2- Apply the novel probability axioms and paradigm to Buffon's needle method for the computation of π.

3- Show that all nondeterministic phenomena can be expressed deterministically in the complex probabilities set which is $\boldsymbol{\mathcal{C}}$.

4- Compute and quantify both the degree of our knowledge and the chaotic factor of Buffon's needle procedure.

5- Represent and show the graphs of the functions and parameters of the innovative paradigm related to Buffon's needle algorithm.

6- Demonstrate that the classical concept of probability is permanently equal to one in the set of complex probabilities; hence, no chaos, no randomness, no ignorance, no uncertainty, no unpredictability, no nondeterminism, and no disorder exist in:

$$\boldsymbol{\mathcal{C}} \text{ (complex set)} = \boldsymbol{\mathcal{R}} \text{ (real set)} + \boldsymbol{\mathcal{M}} \text{ (imaginary set)}.$$

7- Pave the way to implement this inventive model to other topics in prognostics and to the field of stochastic processes. These will be the goals of my future research works.

Concerning some applications of the novel established paradigm and as a future work, it can be applied to any nondeterministic phenomena using Buffon's needle algorithm in any random case.

Moreover, compared with existing literature, the major contribution of the current research chapter is to apply the innovative paradigm of complex probability to the concept and technique of the probabilistic Buffon's needle simulation and algorithms. The next figure displays the major aims and purposes of the Complex Probability Paradigm (*CPP*) (Figure 1).

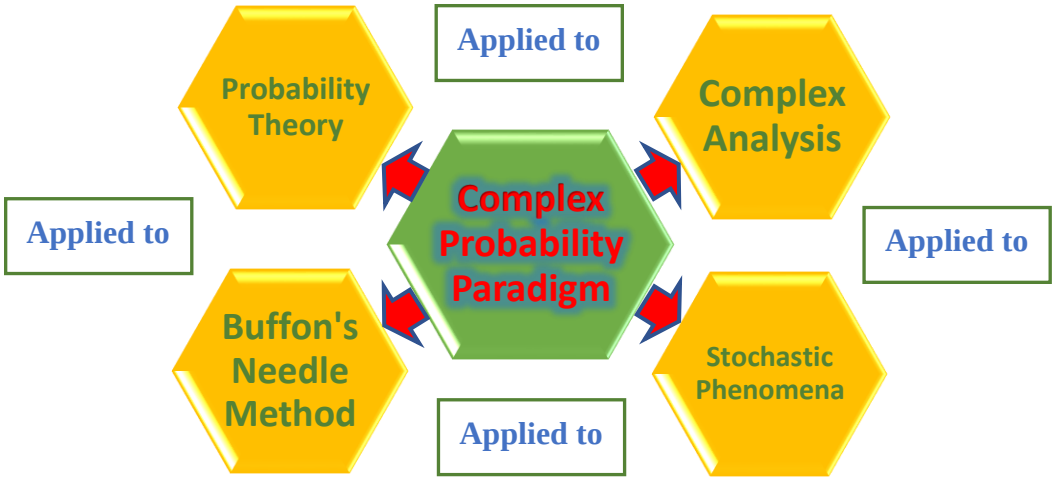


Figure 1- The diagram of the major aims of the Complex Probability Paradigm and Buffon's Needle Method.

3- The Complex Probability Paradigm [37-142]

3-1- The Original Andrey Nikolaevich Kolmogorov System of Axioms

The simplicity of Kolmogorov's system of axioms may be surprising. Let E be a collection of elements $\{E_1, E_2, \ldots\}$ called elementary events and let F be a set of subsets of E called random events. The five axioms for a finite set E are:

Axiom 1: F is a field of sets.
Axiom 2: F contains the set E.
Axiom 3: A non-negative real number $P_{rob}(A)$, called the probability of A, is assigned to each set A in F. We have always $0 \le P_{rob}(A) \le 1$.
Axiom 4: $P_{rob}(E)$ equals 1.
Axiom 5: If A and B have no elements in common, the number assigned to their union is:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B)$$

hence, we say that A and B are disjoint; otherwise, we have:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B) - P_{rob}(A \cap B)$$

And we say also that: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B/A) = P_{rob}(B) \times P_{rob}(A/B)$ which is the conditional probability. If both A and B are independent then: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B)$.

Moreover, we can generalize and say that for N disjoint (mutually exclusive) events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following additivity rule:

$$P_{rob}\left(\bigcup_{j=1}^{N} A_j\right) = \sum_{j=1}^{N} P_{rob}\left(A_j\right)$$

And we say also that for *N* independent events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following product rule:

$$P_{rob}\left(\bigcap_{j=1}^{N} A_j\right) = \prod_{j=1}^{N} P_{rob}\left(A_j\right)$$

3-2- Adding the Imaginary Part $\mathcal{M}$

Now, we can add to this system of axioms an imaginary part such that:

Axiom 6: Let $P_m = i \times (1 - P_r)$ be the probability of an associated complementary event in $\mathcal{M}$ (the imaginary part) to the event *A* in $\mathcal{R}$ (the real part). It follows that $P_r + P_m / i = 1$ where *i* is the imaginary number with $i = \sqrt{-1}$ or $i^2 = -1$.

Axiom 7: We construct the complex number or vector $Z = P_r + P_m = P_r + i(1 - P_r)$ having a norm $|Z|$ such that:

$$|Z|^2 = P_r^2 + (P_m / i)^2 .$$

Axiom 8: Let *Pc* denote the probability of an event in the complex probability universe $\mathcal{C}$ where $\mathcal{C} = \mathcal{R} + \mathcal{M}$. We say that *Pc* is the probability of an event *A* in $\mathcal{R}$ with its associated event in $\mathcal{M}$ such that:

$$Pc^2 = (P_r + P_m / i)^2 = |Z|^2 - 2iP_rP_m \text{ and is always equal to 1.}$$

We can see that by taking into consideration the set of imaginary probabilities we added three new and original axioms and consequently the system of axioms defined by Kolmogorov was hence expanded to encompass the set of imaginary numbers.

3-3- A Brief Interpretation of the Novel Paradigm

To summarize and to conclude, as the degree of our certain knowledge in the real universe $\mathcal{R}$ is unfortunately incomplete, the extension to the complex set $\mathcal{C}$ includes the contributions of both the real set of probabilities $\mathcal{R}$ and the imaginary set of probabilities $\mathcal{M}$. Consequently, this will result in a complete and perfect degree of knowledge in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ (since *Pc* = 1). In fact, in order to have a certain prediction of any random event, it is necessary to work in the complex set $\mathcal{C}$ in which the chaotic factor is quantified and subtracted from the computed degree of knowledge to lead to a probability in $\mathcal{C}$ equal to one as it is shown in the following equation derived from *CPP*:

$$Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc.$$

This hypothesis is also verified in my previous research papers and works by the mean of many examples encompassing both discrete and continuous distributions. The Extended Kolmogorov Axioms (*EKA* for short) or the Complex Probability Paradigm (*CPP* for short) can be illustrated by the following figure (Figure 2):

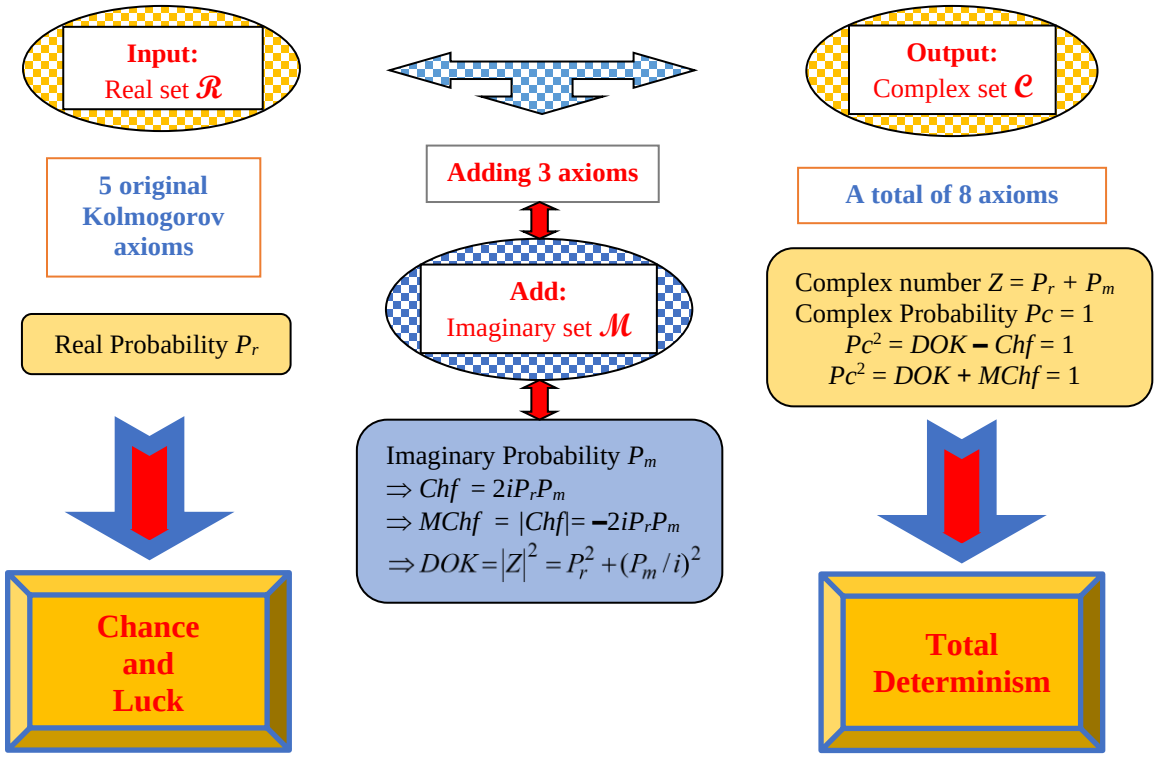


Figure 2- The *EKA* or the *CPP* diagram

4- The Buffon's Needle Method for the computation of π

A classic example of what we call the Monté Carlo method is that of *Georges-Louis Leclerc* or *Comte de Buffon*, who in 1733 pointed out that π could be determined experimentally by repeatedly throwing a needle onto a ruled surface and counting the number of times the needle crossed a line. The idea is more remarkable for its sophistication in geometric probability than for its practicability – a more accurate evaluation of π could be done with a piece of string, a ruler, and the plates and saucers in your kitchen. But the idea of Monté Carlo had been conceived, although the difficulty of using physical devices for sampling and the lack of suitable statistical theory made it little more than a curiosity until the advent of large-scale computers.

The Buffon's needle solution: If a needle of length L (≤ 1) is dropped on a ruled surface of parallel lines spaced one unit apart (Figure 3), the probability that the needle will cross a line is $\frac{2L}{\pi}$. If the needle is dropped N times, the number of line crossings (say, X) should be about $\frac{2NL}{\pi}$, and hence:

$\frac{2NL}{X}$ is a Monté Carlo estimate of π

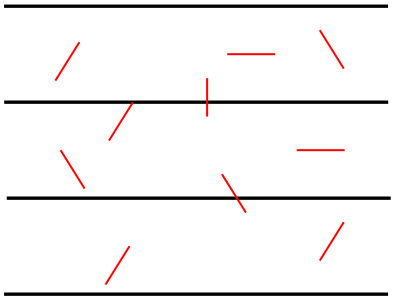


Figure 3- Needles on a ruled surface.

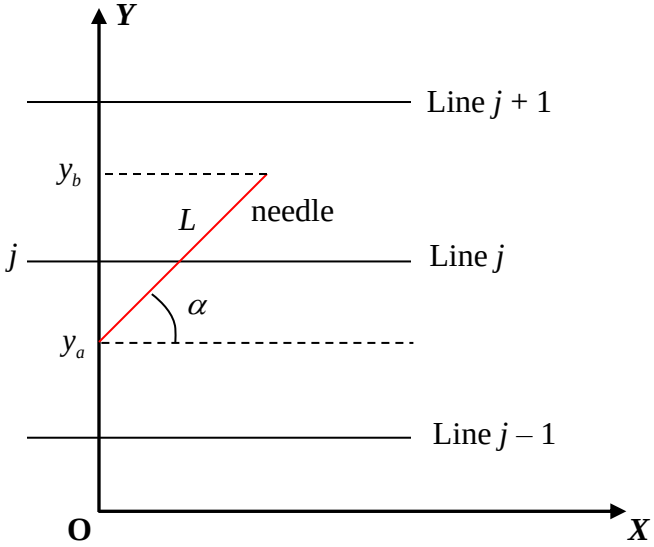


Figure 4- Geometrical explanation of the Buffon's needle problem.

We have y_a a random number and α is a random angle.

We take $y_b = y_a + \sin\alpha$, and L = length of the needle = 1.

Let $q_1 = |j - y_a|$ and $q_2 = |y_b - y_a|$,

If $(q_1 < q_2)$ *then* there is a line crossing,

else, the needle doesn't cross the line (Figure 4).

Inside the computer program, the variable *counter* counts the number of line crossings, and the variable N = number of iterations such that: $0 \le N \le N_C$; therefore:

Exact Result = $EX = \pi = 3.1415926535897931\ldots$

Approximate Result = $\dfrac{2N_C L}{X} = \dfrac{2 \times N_C \times (L=1)}{counter}$ = the variable *AP* in the whole work.

5- The Buffon's Needle Method and the Complex Probability Paradigm Parameters

5-1- The Convergence and Divergence Probabilities

Let EX be the experiment exact result like of a multidimensional or a simple integral that is not possible always to determine by ordinary methods or calculus or numerical deterministic methods. And let AP be the experiment approximate result and therefore let it be the value of these random experiments found by Monte Carlo Techniques like Buffon's needle method.

The absolute error in the numerical analysis method is: $\text{Abs. Error} = |EX - AP|$

The relative error in the numerical method is: $\text{Rel. Error} = \left|\frac{\text{Abs. Error}}{EX}\right| = \left|\frac{EX - AP}{EX}\right| = \left|1 - \frac{AP}{EX}\right|$

In addition, the percent relative error is = $100\% \times \left|\frac{EX - AP}{EX}\right|$ and is always between 0% and 100%.

Therefore, the relative error is always between 0 and 1. Hence:

$$0 \le \left|\frac{EX - AP}{EX}\right| \le 1 \Leftrightarrow \begin{cases} 0 \le \left(\frac{EX - AP}{EX}\right) \le 1 & \text{if } AP \le EX \\ 0 \le -\left(\frac{EX - AP}{EX}\right) \le 1 & \text{if } AP \ge EX \end{cases} \Leftrightarrow \begin{cases} 0 \le AP \le EX \\ EX \le AP \le 2EX \end{cases}$$

Moreover, we define the real probability by:

$$P_r = 1 - \left|\frac{EX - AP}{EX}\right| = 1 - \left|1 - \frac{AP}{EX}\right| = \begin{cases} 1 - \left(1 - \frac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ 1 + \left(1 - \frac{AP}{EX}\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} \frac{AP}{EX} & \text{if } 0 \le AP \le EX \\ 2 - \frac{AP}{EX} & \text{if } EX \le AP \le 2EX \end{cases}$$

= 1 – the relative error in the numerical method
= Probability and degree of the numerical method convergence in $\mathbb{R}$
= Probability and ratio of the approximate result to the exact result.

And therefore:

$$P_m = i\left(1-P_r\right) = i\left\{1-\left[1-\left|\frac{EX-AP}{EX}\right|\right]\right\} = i\left\{1-\left[1-\left|1-\frac{AP}{EX}\right|\right]\right\} = i\left|1-\frac{AP}{EX}\right|$$

$$= \begin{cases} i\left(1-\dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ -i\left(1-\dfrac{AP}{EX}\right) & \text{if } EX \le AP \le 2EX \end{cases} = \begin{cases} i\left(1-\dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ i\left(\dfrac{AP}{EX}-1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

= The relative error in the numerical method in $\mathcal{M}$

= Probability and degree of the numerical method divergence in the imaginary probability set $\mathcal{M}$ since it is the imaginary complement of P_r.

Consequently,

$$P_m / i = 1-P_r = \left|1-\frac{AP}{EX}\right| = \begin{cases} 1-\dfrac{AP}{EX} & \text{if } 0 \le AP \le EX \\ \dfrac{AP}{EX}-1 & \text{if } EX \le AP \le 2EX \end{cases}$$

= The relative error in the numerical method in $\mathcal{R}$

= Probability and degree of the numerical method divergence in $\mathcal{R}$ since it is the real complement of P_r.

In the case where $0 \le AP \le EX$ we have $0 \le \frac{AP}{EX} \le 1 \Rightarrow 0 \le P_r \le 1$ and we deduce also that $0 \le \left(1-\frac{AP}{EX}\right) \le 1 \Rightarrow 0 \le P_m / i \le 1$ and $\Rightarrow 0 \le P_m \le i$

And in the case where $EX \le AP \le 2EX \Rightarrow 1 \le \frac{AP}{EX} \le 2 \Rightarrow 0 \le \left(2-\frac{AP}{EX}\right) \le 1 \Rightarrow 0 \le P_r \le 1$ and we deduce also that $0 \le \left(\frac{AP}{EX}-1\right) \le 1 \Rightarrow 0 \le P_m / i \le 1$ and $\Rightarrow 0 \le P_m \le i$

Therefore, if $AP = 0$ or $AP = 2EX$ that means before the beginning of the numerical method and the simulation, then:

$$P_r = P_{rob}\,(\text{convergence}) \text{ in } \mathcal{R} = 0$$
$$P_m = P_{rob}\,(\text{divergence}) \text{ in } \mathcal{M} = i$$
$$P_m / i = P_{rob}\,(\text{divergence}) \text{ in } \mathcal{R} = 1$$

And if $AP = EX$ that means at the end of the simulation and the numerical method then:

$$P_r = P_{rob}\,(\text{convergence}) \text{ in } \mathcal{R} = 1$$
$$P_m = P_{rob}\,(\text{divergence}) \text{ in } \mathcal{M} = 0$$
$$P_m / i = P_{rob}\,(\text{divergence}) \text{ in } \mathcal{R} = 0$$

5-2- The Complex Random and Random Vector Z in $\mathcal{C} = \mathcal{R} + \mathcal{M}$

We have $Z = P_r + P_m = \begin{cases} \dfrac{AP}{EX} + i\left(1 - \dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ \left(2 - \dfrac{AP}{EX}\right) + i\left(\dfrac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases}$

where $\mathrm{Re}(Z) = P_r = \begin{cases} \dfrac{AP}{EX} & \text{if } 0 \le AP \le EX \\ 2 - \dfrac{AP}{EX} & \text{if } EX \le AP \le 2EX \end{cases}$ = the real part of Z

and $\mathrm{Im}(Z) = P_m / i = \begin{cases} 1 - \dfrac{AP}{EX} & \text{if } 0 \le AP \le EX \\ \dfrac{AP}{EX} - 1 & \text{if } EX \le AP \le 2EX \end{cases}$ = the imaginary part of Z.

That means that the complex random vector Z is the sum in $\mathcal{C}$ of the convergence real probability in $\mathcal{R}$ and of the divergence imaginary probability in $\mathcal{M}$.

If $AP = 0$ or $AP = 2EX$ (before beginning of the simulation) then $P_r = \dfrac{AP}{EX} = \dfrac{0}{EX} = 0$ or $P_r = 2 - \dfrac{AP}{EX} = 2 - \dfrac{2EX}{EX} = 2 - 2 = 0$ and $P_m = i\left(1 - \dfrac{AP}{EX}\right) = i\left(1 - \dfrac{0}{EX}\right) = i(1-0) = i$ or $P_m = i\left(\dfrac{AP}{EX} - 1\right) = i\left(\dfrac{2EX}{EX} - 1\right) = i(2-1) = i$ therefore $Z = 0 + i = i$.

If $AP = \dfrac{EX}{2}$ or $AP = \dfrac{3EX}{2}$ (at the middle of the simulation) then:

$$P_r = \begin{cases} \dfrac{AP}{EX} & \text{if } 0 \le AP \le EX \\ 2 - \dfrac{AP}{EX} & \text{if } EX \le AP \le 2EX \end{cases} = \begin{cases} \dfrac{EX}{2EX} = 0.5 & \text{if } 0 \le AP \le EX \\ 2 - \dfrac{3EX}{2EX} = 0.5 & \text{if } EX \le AP \le 2EX \end{cases}$$

$\Leftrightarrow P_r = 0.5$

$$\text{and } P_m = \begin{cases} i\left(1 - \dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ i\left(\dfrac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases} = \begin{cases} i\left(1 - \dfrac{EX}{2EX}\right) = 0.5i & \text{if } 0 \le AP \le EX \\ i\left(\dfrac{3EX}{2EX} - 1\right) = 0.5i & \text{if } EX \le AP \le 2EX \end{cases}$$

$\Leftrightarrow P_m = 0.5i$

therefore $Z = 0.5 + 0.5i$.

If $AP = EX$ (at the simulation end) then:

$$P_r=\begin{cases}\dfrac{AP}{EX}=\dfrac{EX}{EX}=1 & \text{if } 0\le AP\le EX\\ 2-\dfrac{AP}{EX}=2-\dfrac{EX}{EX}=2-1=1 & \text{if } EX\le AP\le 2EX\end{cases}\Leftrightarrow P_r=1$$

And

$$P_m=\begin{cases}i\left(1-\dfrac{AP}{EX}\right) & \text{if } 0\le AP\le EX\\ i\left(\dfrac{AP}{EX}-1\right) & \text{if } EX\le AP\le 2EX\end{cases}=\begin{cases}i\left(1-\dfrac{EX}{EX}\right) & \text{if } 0\le AP\le EX\\ i\left(\dfrac{EX}{EX}-1\right) & \text{if } EX\le AP\le 2EX\end{cases}$$

$$=\begin{cases}0 & \text{if } 0\le AP\le EX\\ 0 & \text{if } EX\le AP\le 2EX\end{cases}$$

$\Leftrightarrow P_m=0$

therefore $Z=1+0i=1$.

5-3- The Degree of Our Knowledge of the Random Experiment *DOK*

We have:

$$DOK=|Z|^2=P_r^2+(P_m/i)^2$$

$$=\begin{cases}\left(\dfrac{AP}{EX}\right)^2 & \text{if } 0\le AP\le EX\\ \left(2-\dfrac{AP}{EX}\right)^2 & \text{if } EX\le AP\le 2EX\end{cases}+\begin{cases}\left(1-\dfrac{AP}{EX}\right)^2 & \text{if } 0\le AP\le EX\\ \left(\dfrac{AP}{EX}-1\right)^2 & \text{if } EX\le AP\le 2EX\end{cases}$$

$$=\begin{cases}\left(\dfrac{AP}{EX}\right)^2+\left(1-\dfrac{AP}{EX}\right)^2 & \text{if } 0\le AP\le EX\\ \left(2-\dfrac{AP}{EX}\right)^2+\left(\dfrac{AP}{EX}-1\right)^2 & \text{if } EX\le AP\le 2EX\end{cases}$$

$$=\begin{cases}2\left(\dfrac{AP}{EX}\right)^2-2\left(\dfrac{AP}{EX}\right)+1 & \text{if } 0\le AP\le EX\\ 2\left(\dfrac{AP}{EX}\right)^2-6\left(\dfrac{AP}{EX}\right)+5 & \text{if } EX\le AP\le 2EX\end{cases}$$

From *CPP* we have that $0.5\le DOK\le 1$ then if $DOK=0.5$

$$\Leftrightarrow\begin{cases}2\left(\dfrac{AP}{EX}\right)^2-2\left(\dfrac{AP}{EX}\right)+1=0.5 & \text{if } 0\le AP\le EX\\ 2\left(\dfrac{AP}{EX}\right)^2-6\left(\dfrac{AP}{EX}\right)+5=0.5 & \text{if } EX\le AP\le 2EX\end{cases}$$

then solving the two second-degree equations for $\frac{AP}{EX}$ gives:

$$\begin{cases} \frac{AP}{EX} = 1/2 & \text{if } 0 \le AP \le EX \\ \frac{AP}{EX} = 3/2 & \text{if } EX \le AP \le 2EX \end{cases} \Leftrightarrow \begin{cases} AP = EX/2 & \text{if } 0 \le AP \le EX \\ AP = 3EX/2 & \text{if } EX \le AP \le 2EX \end{cases}$$

and vice versa.

That means that *DOK* is minimum when the approximate result is equal to half of the exact result if $0 \le AP \le EX$ or when the approximate result is equal to three times the half of the exact result if $EX \le AP \le 2EX$, that means at the middle of the simulation.

In addition, if $DOK = 1$ then:

$$\Leftrightarrow \begin{cases} 2\left(\frac{AP}{EX}\right)^2 - 2\left(\frac{AP}{EX}\right) + 1 = 1 & \text{if } 0 \le AP \le EX \\ 2\left(\frac{AP}{EX}\right)^2 - 6\left(\frac{AP}{EX}\right) + 5 = 1 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(\frac{AP}{EX}\right)^2 - \left(\frac{AP}{EX}\right) = 0 & \text{if } 0 \le AP \le EX \\ 2\left(\frac{AP}{EX}\right)^2 - 6\left(\frac{AP}{EX}\right) + 4 = 0 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} AP = 0 \text{ OR } AP = EX & \text{if } 0 \le AP \le EX \\ AP = 2EX \text{ OR } AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$$

and vice versa.

That means that *DOK*, which is the degree of our knowledge of the random experiment, is maximum and is equal to 1 when the approximate result is equal to EX that means when it is equal to the exact result (at the end of the simulation) or 0 or $2EX$ (before the beginning of the simulation). We can deduce that we have total and perfect knowledge of the random experiment at the end of the simulation after the convergence of the numerical method to the exact result and hence when relative error is 0 = 0% and as well as before the beginning of the simulation since no randomness was introduced yet and thus when relative error is 1 = 100%.

5-4- The Chaotic Factor *Chf*

We have:

$$Chf = 2iP_rP_m = 2i \times \begin{cases} \dfrac{AP}{EX} & \text{if } 0 \le AP \le EX \\ 2 - \dfrac{AP}{EX} & \text{if } EX \le AP \le 2EX \end{cases} \times \begin{cases} i\left(1 - \dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ i\left(\dfrac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

since $i^2 = -1$ then:

$$Chf = \begin{cases} -2\left(\dfrac{AP}{EX}\right)\left(1 - \dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ -2\left(2 - \dfrac{AP}{EX}\right)\left(\dfrac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

From *CPP* we have that $-0.5 \le Chf \le 0$ then if $Chf = -0.5$

$$\Leftrightarrow \begin{cases} -2\left(\dfrac{AP}{EX}\right)\left(1 - \dfrac{AP}{EX}\right) = -0.5 & \text{if } 0 \le AP \le EX \\ -2\left(2 - \dfrac{AP}{EX}\right)\left(\dfrac{AP}{EX} - 1\right) = -0.5 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} AP = EX / 2 & \text{if } 0 \le AP \le EX \\ AP = 3EX / 2 & \text{if } EX \le AP \le 2EX \end{cases}$$

and vice versa.

That means that *Chf* is minimum when the approximate result is equal to half of the exact result if $0 \le AP \le EX$ or when the approximate result is equal to three times the half of the exact result if $EX \le AP \le 2EX$, that means at the middle of the simulation.
In addition, if $Chf = 0$ then:

$$\Leftrightarrow \begin{cases} -2\left(\dfrac{AP}{EX}\right)\left(1 - \dfrac{AP}{EX}\right) = 0 & \text{if } 0 \le AP \le EX \\ -2\left(2 - \dfrac{AP}{EX}\right)\left(\dfrac{AP}{EX} - 1\right) = 0 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} AP = 0 \text{ OR } AP = EX & \text{if } 0 \le AP \le EX \\ AP = 2EX \text{ OR } AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$$

And, conversely, if $\begin{cases} AP = 0 \text{ OR } AP = EX & \text{if } 0 \le AP \le EX \\ AP = 2EX \text{ OR } AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$ then $Chf = 0$.

That means that *Chf* is equal to 0 when the approximate result is equal to EX that means when it is equal to the exact result (at the end of the simulation) or 0 or $2EX$ (before the beginning of the simulation).

5-5- The Magnitude of the Chaotic Factor *MChf*

We have:

$$MChf = |Chf| = -2iP_rP_m$$

$$= -2i \times \begin{cases} \dfrac{AP}{EX} & \text{if } 0 \le AP \le EX \\ 2 - \dfrac{AP}{EX} & \text{if } EX \le AP \le 2EX \end{cases} \times \begin{cases} i\left(1 - \dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ i\left(\dfrac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

since $i^2 = -1$ then:

$$MChf = \begin{cases} 2\left(\dfrac{AP}{EX}\right)\left(1 - \dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ 2\left(2 - \dfrac{AP}{EX}\right)\left(\dfrac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

From *CPP* we have that $0 \le MChf \le 0.5$ then if $MChf = 0.5$

$$\Leftrightarrow \begin{cases} 2\left(\dfrac{AP}{EX}\right)\left(1 - \dfrac{AP}{EX}\right) = 0.5 & \text{if } 0 \le AP \le EX \\ 2\left(2 - \dfrac{AP}{EX}\right)\left(\dfrac{AP}{EX} - 1\right) = 0.5 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} AP = EX/2 & \text{if } 0 \le AP \le EX \\ AP = 3EX/2 & \text{if } EX \le AP \le 2EX \end{cases}$$

and vice versa.

That means that *MChf* is maximum when the approximate result is equal to half of the exact result if $0 \le AP \le EX$ or when the approximate result is equal to three times the half of the exact result if $EX \le AP \le 2EX$, that means at the middle of the simulation. This implies that the magnitude of the chaos (*MChf*) introduced by the variables used in the numerical method is maximum at the halfway of the simulation.

In addition, if $MChf = 0$ then:

$$\Leftrightarrow \begin{cases} 2\left(\dfrac{AP}{EX}\right)\left(1 - \dfrac{AP}{EX}\right) = 0 & \text{if } 0 \le AP \le EX \\ 2\left(2 - \dfrac{AP}{EX}\right)\left(\dfrac{AP}{EX} - 1\right) = 0 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} AP = 0 \text{ OR } AP = EX & \text{if } 0 \le AP \le EX \\ AP = 2EX \text{ OR } AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$$

And, conversely, if $\begin{cases} AP = 0 \text{ OR } AP = EX & \text{if } 0 \le AP \le EX \\ AP = 2EX \text{ OR } AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$ then $MChf = 0$.

That means that *MChf* is minimum and is equal to 0 when the approximate result is equal to EX that means when it is equal to the exact result (at the end of the simulation) or 0 or $2EX$ (before the beginning of the simulation). We can deduce that the magnitude of the chaos in the random experiment is null at the end of the simulation after the convergence of the numerical method to the exact result and when randomness has finished its task in the numerical method and experiment as well as before the beginning of the simulation since no randomness was introduced yet.

5-6- The Probability *Pc* in the Probability Set $\mathcal{C} = \mathcal{R} + \mathcal{M}$

We have from *CPP*:

$$Pc^2 = DOK - Chf = DOK + MChf$$

$$= \begin{cases} 2\left(\frac{AP}{EX}\right)^2 - 2\left(\frac{AP}{EX}\right) + 1 & \text{if } 0 \le AP \le EX \\ 2\left(\frac{AP}{EX}\right)^2 - 6\left(\frac{AP}{EX}\right) + 5 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$- \begin{cases} -2\left(\frac{AP}{EX}\right)\left(1 - \frac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ -2\left(2 - \frac{AP}{EX}\right)\left(\frac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} 1 & \text{if } 0 \le AP \le EX \\ 1 & \text{if } EX \le AP \le 2EX \end{cases} \Leftrightarrow Pc^2 = 1 \quad \text{for } 0 \le AP \le 2EX$$

$\Leftrightarrow Pc = 1$ = Probability and degree of convergence in $\mathcal{C}$, therefore:

$$Pc = \begin{cases} \frac{AP}{EX} = 1 & \text{if } 0 \le AP \le EX \\ 2 - \frac{AP}{EX} = 1 & \text{if } EX \le AP \le 2EX \end{cases} \Leftrightarrow \begin{cases} AP = EX & \text{if } 0 \le AP \le EX \\ AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$$

$\Leftrightarrow AP = EX$ for $0 \le AP \le 2EX$ continuously in the probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$. This is due to the fact that in $\mathcal{C}$ we have subtracted in the equation above from our knowledge *DOK* the chaotic factor *Chf* and consequently we have removed chaos introduced and caused by all the variables and the numerical fluctuations that lead to approximate results in the numerical simulation in $\mathcal{R}$. Therefore, since in $\mathcal{C}$ we have always $AP = EX$ then the simulation which is a random method by nature in $\mathcal{R}$ becomes after applying the *CPP* a non-random method in $\mathcal{C}$ since the convergence probability of any experiment in $\mathcal{C}$ is permanently and constantly equal to 1 for any subintervals or iterations number *N*.

5-7- The Rates of Change of the Probabilities in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$

$$\text{Since } Z = P_r + P_m = \begin{cases} \dfrac{AP}{EX} + i\left(1 - \dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ \left(2 - \dfrac{AP}{EX}\right) + i\left(\dfrac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases} = \mathrm{Re}(Z) + i\,\mathrm{Im}(Z)$$

Then:

$$\frac{dZ}{d(AP)} = \frac{dP_r}{d(AP)} + \frac{dP_m}{d(AP)} = \begin{cases} \dfrac{d}{d(AP)}\left[\dfrac{AP}{EX} + i\left(1 - \dfrac{AP}{EX}\right)\right] & \text{if } 0 \le AP \le EX \\ \dfrac{d}{d(AP)}\left[\left(2 - \dfrac{AP}{EX}\right) + i\left(\dfrac{AP}{EX} - 1\right)\right] & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} \dfrac{d}{d(AP)}\left[\dfrac{AP}{EX}\right] + \dfrac{d}{d(AP)}\left[i\left(1 - \dfrac{AP}{EX}\right)\right] & \text{if } 0 \le AP \le EX \\ \dfrac{d}{d(AP)}\left[2 - \dfrac{AP}{EX}\right] + \dfrac{d}{d(AP)}\left[i\left(\dfrac{AP}{EX} - 1\right)\right] & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} \dfrac{1}{EX} - \dfrac{i}{EX} = \dfrac{1}{EX}(1 - i) & \text{if } 0 \le AP \le EX \\ -\dfrac{1}{EX} + \dfrac{i}{EX} = \dfrac{1}{EX}(i - 1) & \text{if } EX \le AP \le 2EX \end{cases}$$

Therefore,

$$\mathrm{Re}\left[\frac{dZ}{d(AP)}\right] = \frac{dP_r}{d(AP)} = \begin{cases} +\dfrac{1}{EX} & \text{if } 0 \le AP \le EX \\ -\dfrac{1}{EX} & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} \text{constant} > 0 & \text{if } 0 \le AP \le EX \text{ and } EX > 0 \\ \text{constant} < 0 & \text{if } EX \le AP \le 2EX \text{ and } EX > 0 \end{cases}$$

that means that the rate of change or the slope of the probability of convergence in $\mathcal{R}$ is positive and constant if $0 \le AP \le EX$, and negative and constant if $EX \le AP \le 2EX$, and it depends only on $EX > 0$; hence, we have a constant increase in P_r (the convergence probability which is by definition an absolute value that means always nonnegative) as a function of the iterations or subintervals number *N* as AP increases from 0 to EX and as AP decreases from $2EX$ to EX till P_r reaches the value 1 that means till the random experiment converges to EX .

And $$= \begin{cases} \text{constant} < 0 & \text{if } EX \le AP \le 0 \text{ and } EX < 0 \\ \text{constant} > 0 & \text{if } 2EX \le AP \le EX \text{ and } EX < 0 \end{cases}$$

that means that the slope of the probability of convergence in $\mathcal{R}$ or its rate of change is constant and negative if $EX \le AP \le 0$, and constant and positive $2EX \le AP \le EX$, and it depends only on $EX < 0$; hence, we have a constant increase in P_r as a function of the iterations or subintervals number *N* as AP decreases from 0 to EX and as AP increases from $2EX$ to EX till P_r reaches the value 1 that means till the random experiment converges to EX .

$$\text{Im}\left[\frac{dZ}{d(AP)}\right] = \frac{1}{i}\frac{dP_m}{d(AP)} = \frac{d(P_m/i)}{d(AP)} = \begin{cases} -\dfrac{1}{EX} & \text{if } 0 \le AP \le EX \\ +\dfrac{1}{EX} & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} \text{constant} < 0 & \text{if } 0 \le AP \le EX \text{ and } EX > 0 \\ \text{constant} > 0 & \text{if } EX \le AP \le 2EX \text{ and } EX > 0 \end{cases}$$

that means that their rates of change or the slopes of the probabilities of divergence in $\mathcal{R}$ and $\mathcal{M}$ are negative and constant if $0 \le AP \le EX$, and positive and constant if $EX \le AP \le 2EX$, and they depend only on $EX > 0$; hence, we have a constant decrease in P_m/i and P_m (the divergence probabilities) as functions of the iterations or subintervals number *N* as AP increases from 0 to EX and as AP decreases from $2EX$ to EX till P_m/i and P_m reach the value 0 that means till the random experiment converges to EX .

And $$= \begin{cases} \text{constant} > 0 & \text{if } EX \le AP \le 0 \text{ and } EX < 0 \\ \text{constant} < 0 & \text{if } 2EX \le AP \le EX \text{ and } EX < 0 \end{cases}$$

that means that the slopes of the probabilities of divergence in $\mathcal{R}$ and $\mathcal{M}$ or their rates of change are constant and positive if $EX \le AP \le 0$, and constant and negative if $2EX \le AP \le EX$, and they depend only on $EX < 0$; hence, we have a constant decrease in P_m/i and P_m as functions of the iterations or subintervals number *N* as AP decreases from 0 to EX and as AP increases from $2EX$ to EX till P_m/i and P_m reach the value 0 that means till the random experiment converges to EX .

Additionally,

$$\left|\frac{dZ}{d(AP)}\right|^2=\left[\frac{dP_r}{d(AP)}\right]^2+\left[\frac{1}{i}\times\frac{dP_m}{d(AP)}\right]^2=\left[\frac{dP_r}{d(AP)}\right]^2+\left[\frac{d(P_m/i)}{d(AP)}\right]^2$$

$$=\begin{cases}\left(\frac{1}{EX}\right)^2+\left(-\frac{1}{EX}\right)^2 & \text{if } 0\le AP\le EX\\ \left(-\frac{1}{EX}\right)^2+\left(\frac{1}{EX}\right)^2 & \text{if } EX\le AP\le 2EX\end{cases}$$

$$\Leftrightarrow\left|\frac{dZ}{d(AP)}\right|^2=\frac{1}{(EX)^2}+\frac{1}{(EX)^2}=\frac{2}{(EX)^2}\quad \text{for } 0\le AP\le 2EX$$

$$\Leftrightarrow\left|\frac{dZ}{d(AP)}\right|=\frac{\sqrt{2}}{|EX|}=\text{constant}>0,\ \forall EX\ ;$$

that means that its rate of change or the module of the slope of the complex probability vector Z in $\mathcal{C}$ is positive and constant and it depends only on $|EX|$; hence, we have a constant increase in $\mathrm{Re}(Z)$ and a constant decrease in $\mathrm{Im}(Z)$ as functions of the iterations or subintervals number N and as Z goes from (0, i) at N = 0 till (1,0) at the simulation end; hence, till $\mathrm{Re}(Z)=P_r$ reaches the value 1 that means till the random experiment converges to EX .

Furthermore, since $Pc^2=DOK-Chf=DOK+MChf=1$ from *CPP*

then $Pc=1=$ Probability and degree of convergence in $\mathcal{C}$

and consequently: $\frac{d(Pc)}{d(AP)}=\frac{d(1)}{d(AP)}=0$;

that means that *Pc* is constantly equal to 1 for every value of AP , of EX , and of the iterations or subintervals number *N*, that means for any random experiment and for any simulation of the numerical methods. So, we conclude that in $\mathcal{C}$ we have complete and perfect knowledge of the random experiment which has become now a non-random one since the extension in the complex probability plane $\mathcal{C}$ defined by the *CPP* axioms has changed all random variables to non-random variables and since we have subtracted and eliminated in the equation of *Pc* above chaos expressed by *Chf* from *DOK*. Hence, randomness and chaos vanish completely and totally in the probability set $\mathcal{C}=\mathcal{R}+\mathcal{M}$.

6- The Evaluation of the New Paradigm Parameters

We can deduce from what has been elaborated previously the following:

The real convergence probability: $P_r(N)=1-\left|\frac{EX-AP(N)}{EX}\right|$

We have $0 \le N \le N_C$ where $N = 0$ corresponds to the instant before the beginning of the random experiment when $AP(N=0)=0$ or $=2EX$, and where $N = N_C$ (iterations number needed for the method convergence) corresponds to the instant at the end of the random experiments and Monte Carlo methods when $AP(N=N_C) \to EX$.

The imaginary complementary divergence probability: $P_m(N) = i\left|\frac{EX - AP(N)}{EX}\right|$

The real complementary divergence probability: $P_m(N)/i = \left|\frac{EX - AP(N)}{EX}\right|$

The complex probability and random vector:

$$Z(N) = P_r(N) + P_m(N) = \left[1 - \left|\frac{EX - AP(N)}{EX}\right|\right] + i\left|\frac{EX - AP(N)}{EX}\right|$$

The Degree of Our Knowledge:

$$DOK(N) = |Z(N)|^2 = P_r^2(N) + [P_m(N)/i]^2 = \left[1 - \left|\frac{EX - AP(N)}{EX}\right|\right]^2 + \left[\left|\frac{EX - AP(N)}{EX}\right|\right]^2$$

$$= 1 + 2iP_r(N)P_m(N) = 1 - 2P_r(N)[1 - P_r(N)] = 1 - 2P_r(N) + 2P_r^2(N)$$

$$= 1 - 2\left|\frac{EX - AP(N)}{EX}\right| + 2\left[\frac{EX - AP(N)}{EX}\right]^2$$

$DOK(N)$ is equal to 1 when $P_r(N) = P_r(0) = 0$ and when $P_r(N) = P_r(N_C) = 1$.

The Chaotic Factor:

$$Chf(N) = 2iP_r(N)P_m(N) = -2P_r(N)[1 - P_r(N)] = -2P_r(N) + 2P_r^2(N)$$

$$= -2\left|\frac{EX - AP(N)}{EX}\right| + 2\left[\frac{EX - AP(N)}{EX}\right]^2$$

$Chf(N)$ is null when $P_r(N) = P_r(0) = 0$ and when $P_r(N) = P_r(N_C) = 1$.

The Magnitude of the Chaotic Factor *MChf*:

$$MChf(N) = |Chf(N)| = -2iP_r(N)P_m(N) = 2P_r(N)[1 - P_r(N)] = 2P_r(N) - 2P_r^2(N)$$

$$= 2\left|\frac{EX - AP(N)}{EX}\right| - 2\left[\frac{EX - AP(N)}{EX}\right]^2$$

$MChf(N)$ is null when $P_r(N) = P_r(0) = 0$ and when $P_r(N) = P_r(N_C) = 1$.

At any iteration number *N*: $0 \le \forall N \le N_C$, the probability expressed in the complex probability set $\mathcal{C}$ is the following:

$$Pc^2(N) = [P_r(N) + P_m(N)/i]^2 = |Z(N)|^2 - 2iP_r(N)P_m(N)$$
$$= DOK(N) - Chf(N)$$
$$= DOK(N) + MChf(N)$$
$$= 1$$

then,

$$Pc^2(N) = [P_r(N) + P_m(N)/i]^2 = \{P_r(N) + [1 - P_r(N)]\}^2 = 1^2 = 1 \Leftrightarrow Pc(N) = 1 \text{ always}$$

Hence, the prediction of the convergence probabilities of the stochastic Buffon's needle experiment in the set $\mathcal{C}$ is permanently certain.

Let us consider thereafter a multidimensional integral and a stochastic experiment to simulate the Buffon's needle method and to draw, to visualize, as well as to quantify all the *CPP* and prognostic parameters.

7- The C++ Algorithms of Buffon's Needle Method

7-1- The First Algorithm with the C++ Built-in Uniform Random Number Generator

```
// Buffon's Needle Algorithm with the C++ Built-in Uniform Random
// Number Generator

#include <iostream>
#include <cstdlib>
#include <ctime>
#include <cmath>
#include <iomanip>

using namespace std;
long double total();

const long double PI = 3.1415926535897931;

int main()
{
      long double summation;
      long int c;

      cout << fixed << setprecision(16);

      for (c = 1; c <= 7; c++)
      {
            cout << "THE EXACT VALUE OF PI = " << PI << endl;
            summation = (long double) total();
            cout << "THE ESTIMATION OF PI = " << summation << endl;
```

```
            cout << "THE RELATIVE ERROR = "
                 << 100 * (long double) fabs(1 - summation / PI)
                 << "%\n" << endl;
      }

      return 0;
}

long double total(void)
{
      long int N, Nc, counter = 0;
      long double ya = 0, yb = 0, q1 = 0, q2 = 0, AP = 0, alpha = 0,
                  k = 0, j = 0;

      srand(time(0));

      Nc = 100000000;
      for (N = 1; N <= Nc; N++)
      {
            ya = rand() + (long double) rand() / 32767;
            k = (long double) rand() / 32767;
            alpha = 2 * PI * k;

            yb = ya + (long double) sin(alpha);
            j = (int) fabs(ya) + 1;

            q1 = (long double) fabs(j - ya);
            q2 = (long double) fabs(yb - ya);

            if (q1 < q2)
                  ++counter;

            ya = 0; yb = 0; j = 0; k = 0; alpha = 0; q1 = 0; q2 = 0;
      }

      AP = (long double) 2 * Nc / counter;

      return AP;
}
```

7-2- The Second Algorithm with a Second Uniform Random Number Generator

```
// Buffon's Needle Algorithm with Another Second Uniform Random Number
// Generator

#include <iostream>
```

```
#include <cstdlib>
#include <ctime>
#include <cmath>
#include <iomanip>

using namespace std;

long double generate(long double, long double);
long double total();

const long double PI = 3.1415926535897931;
const long double BOUND = 2147483647;

int main()
{
      long double summation;
      long int c;

      cout << fixed << setprecision(16);

      for (c = 1; c <= 7; c++)
      {
            cout << "THE EXACT VALUE OF PI = " << PI << endl;
            summation = (long double) total();
            cout << "THE ESTIMATION OF PI = " << summation << endl;
            cout << "THE RELATIVE ERROR = "
                 << 100 * (long double) fabs(1 - summation / PI)
                 << "%\n" << endl;
      }

      return 0;
}

long double gen(long double xn1, long double xn2)
{
      long double xn;

      xn = (long double) fmod(((1999 * xn1) + (4444 * xn2)), BOUND);

      return xn;
}

long double total(void)
{
      long int N, Nc, counter = 0;
```

```
        long double ya = 0, yb = 0, q1 = 0, q2 = 0, AP = 0, alpha = 0,
                    k = 0, j = 0, random = 0,
                    sxn1 = 0, sxn2 = 0;

        srand(time(0));
        sxn1 = rand();
        sxn2 = rand();

        Nc = 100000000;
        for (N = 1; N <= Nc; N++)
        {
              random = gen(sxn1, sxn2);
              ya = random;
              sxn2 = sxn1;
              sxn1 = random;
              random = gen(sxn1, sxn2);
              ya = ya + (long double) random / BOUND;

              sxn2 = sxn1;
              sxn1 = random;
              random = gen(sxn1, sxn2);
              k = (long double) random / BOUND;
              alpha = 2 * PI*k;

              yb = ya + (long double) sin(alpha);
              j = (int) fabs(ya) + 1;
              q1 = (long double) fabs(j - ya);
              q2 = (long double) fabs(yb - ya);
              if (q1 < q2)
                    ++counter;

              ya = 0; yb = 0; j = 0; k = 0; alpha = 0; q1 = 0; q2 = 0;
        }

        AP = (long double) 2 * Nc / counter;

        return AP;
}
```

7-3- The Third Algorithm with a Third Uniform Random Number Generator

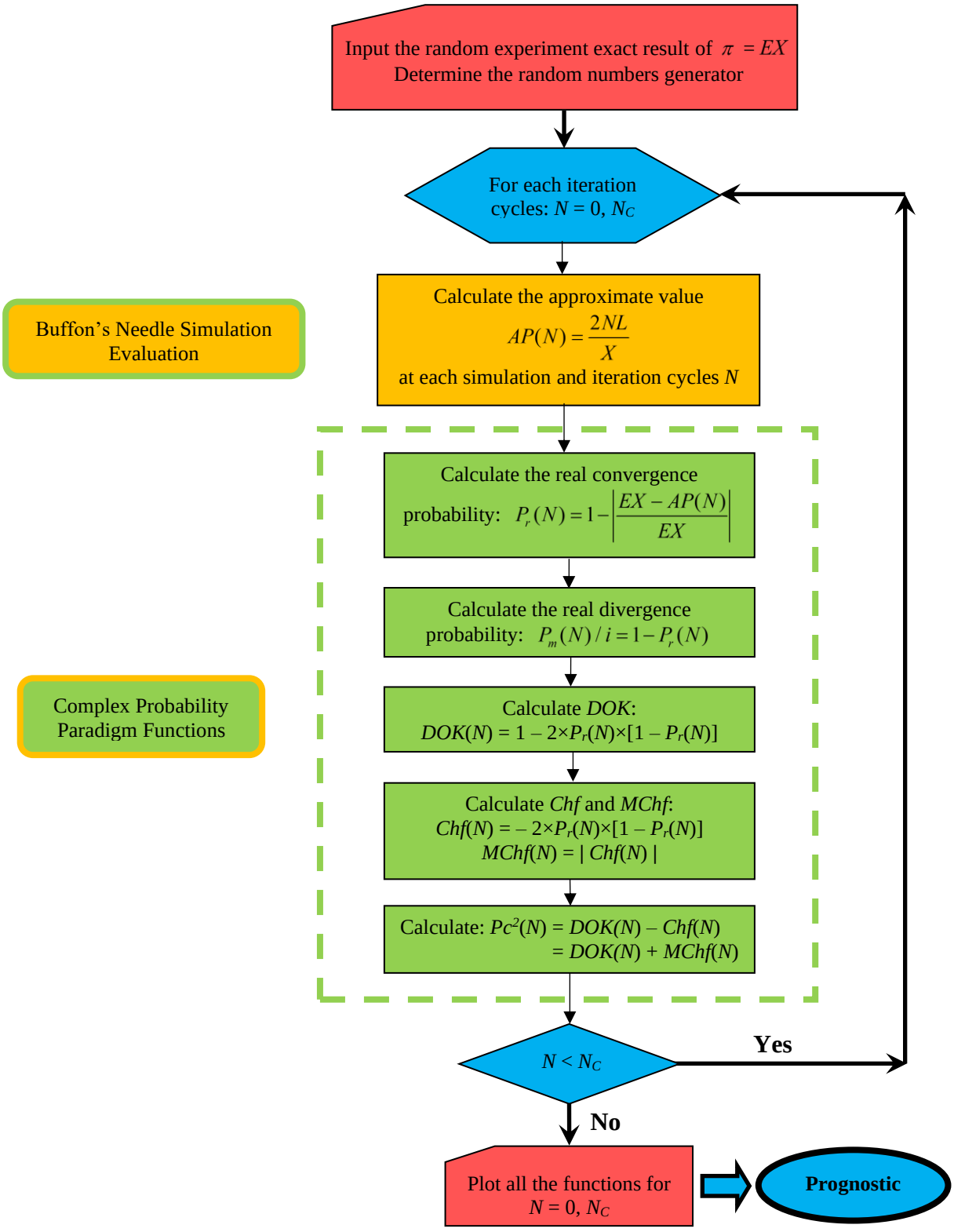
```
// Buffon's Needle Algorithm with Another Third Uniform Random Number
// Generator

#include <iostream>
#include <cstdlib>
```

```
#include <ctime>
#include <cmath>
#include <iomanip>

using namespace std;

long double generate(long double);
long double total();

const long double PI = 3.1415926535897931;
const long double BOUND = 2147483647;

int main()
{
      long double summation;
      long int c;

      cout << fixed << setprecision(16);

      for (c = 1; c <= 7; c++)
      {
            cout << "THE EXACT VALUE OF PI = " << PI << endl;
            summation = (long double) total();
            cout << "THE ESTIMATION OF PI = " << summation << endl;
            cout << "THE RELATIVE ERROR = "
                 << 100 * (long double) fabs(1 - summation / PI)
                 << "%\n" << endl;
      }

      return 0;
}

long double gen(long double xn1)
{
      long double xn;

      xn = (long double) fmod(((69069 * xn1) + 1), BOUND);

      return xn;
}

long double total(void)
{
      long int N, Nc, counter = 0;
      long double ya = 0, yb = 0, q1 = 0, q2 = 0, AP = 0, alpha = 0,
                  k = 0, j = 0, random = 0, sxn1 = 0;
```

```
        srand(time(0));
        sxn1 = rand();

        Nc = 100000000;
        for (N = 1; N <= Nc; N++)
        {
              random = gen(sxn1);
              ya = random;
              sxn1 = random;
              random = gen(sxn1);
              ya = ya + (long double) random / BOUND;

              sxn1 = random;
              random = gen(sxn1);
              k = (long double) random / BOUND;
              alpha = 2 * PI*k;

              yb = ya + (long double) sin(alpha);
              j = (int) fabs(ya) + 1;
              q1 = (long double) fabs(j - ya);
              q2 = (long double) fabs(yb - ya);

              if (q1 < q2)
                    ++counter;

              ya = 0; yb = 0; j = 0; k = 0; alpha = 0; q1 = 0; q2 = 0;
        }

        AP = (long double) 2 * Nc / counter;

        return AP;
  }
```

8- Flowchart of the Complex Probability and Buffon's Needle Technique Prognostic Model

The following flowchart summarizes all the procedures of the proposed complex probability prognostic model:



9- Simulation of the New Paradigm

Note that all the numerical values found in the simulations of the new paradigm for any iteration cycles *N* were computed using the 64-Bit MATLAB version 2024 software. In addition, the reader should take care of the rounding and truncation errors since all numerical values in the computation of π are represented by at most five significant digits and since we are using Buffon's needle method of simulation which gives approximate results subject to random effects and fluctuations. We have considered for this purpose a high-capacity computer system: a workstation computer with parallel microprocessors, a 64-Bit operating system, and a 64-GB RAM. Additionally, we have replaced in all the simulations $AP(N)$ by $AP(N)/2$ and EX by $EX/2=\pi/2$ to better see and read the simulations and to fit all the data and figures in a nicer and improved view.

9-1- The Uniform Random Numbers Generator Case

We will use in the first case in the computation of π using Buffon's needle method the uniform random numbers generator: $(y_a,\alpha,y_b,j,q_1,q_2)\mapsto \mathbb{U}\ (0,10)$

L which is the needle length is taken to be equal to 1.

$\Leftrightarrow AP(N)=\frac{1}{2}\times\frac{2NL}{X}$ with $1\le N\le N_C$ after applying Buffon's needle method.

Moreover, the three figures (Figures 5-7) show the increasing convergence of Buffon's needle method and simulation to the exact result $EX=\pi/2=3.141592654\ldots/2=1.570796327\ldots$ for *N* = 1000, 30000, and $N_C=400000$ iterations. Therefore, we have:

$\lim_{N\to+\infty}P_r(N)=\lim_{N\to+\infty}\left\{1-\left|\frac{EX-AP(N)}{EX}\right|\right\}=1-\left|\frac{EX-EX}{EX}\right|=1-0=1$ which is equal to the convergence probability of Buffon's needle method as $N\to+\infty$.

Additionally, Figure 8 illustrates clearly and visibly the relation of Buffon's needle method to the complex probability paradigm with all its parameters $(Chf, MChf, DOK, EX, AP, P_r, P_m/i, Pc)$ after applying it to this method.

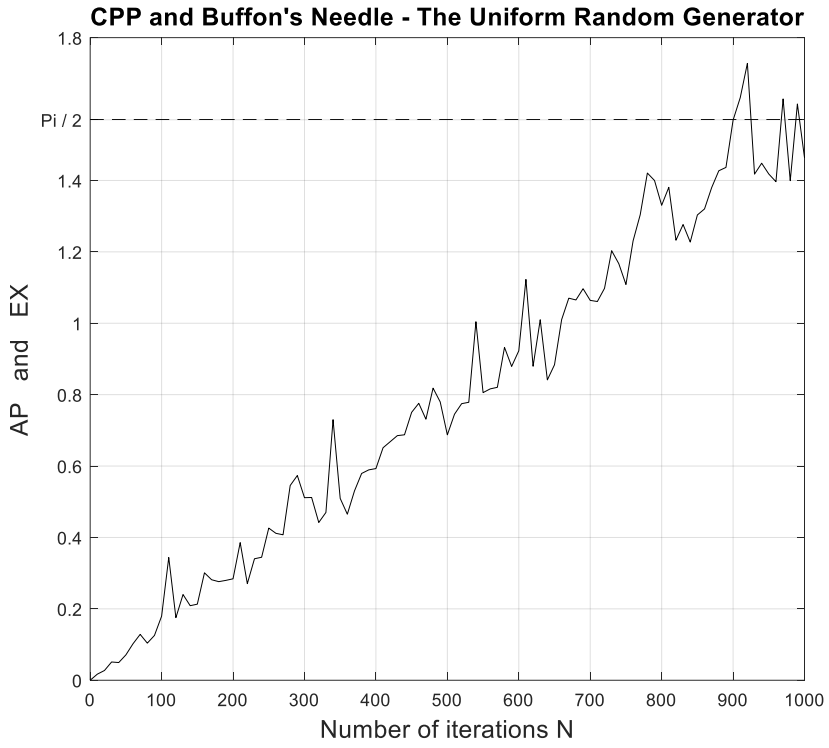


Figure 5- The increasing convergence of Buffon's needle method up to N = 1000 iterations with the Uniform random number generator.

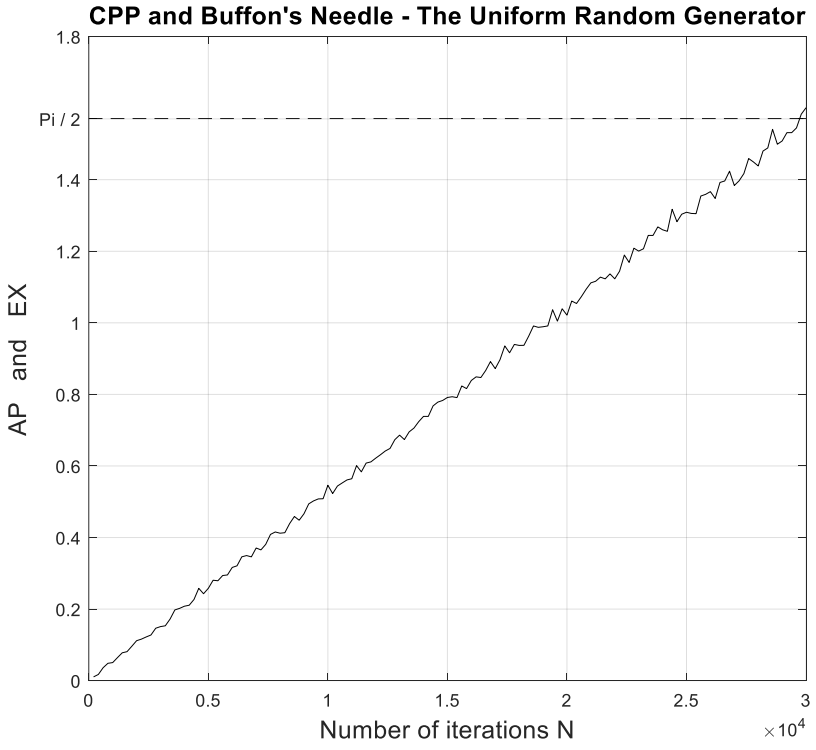


Figure 6- The increasing convergence of Buffon's needle method up to N = 30,000 iterations with the Uniform random number generator.

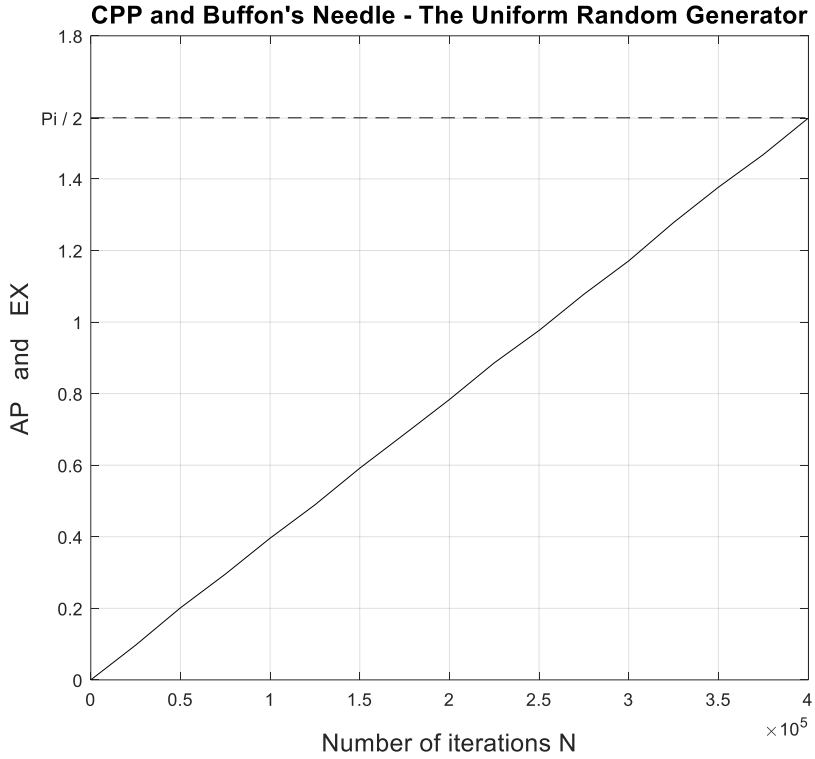


Figure 7- The increasing convergence of Buffon's needle method up to N = 400,000 iterations with the Uniform random number generator.

VII-2-1 The Complex Probability Cubes

In the first cube (Figure 9), the simulation of *DOK* and *Chf* as functions of each other and of the iterations N for the Buffon's needle problem can be seen. The thick line in cyan is the projection of $Pc^2(N) = DOK(N) - Chf(N) = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point J ($DOK = 1$, $Chf = 0$) when $N = 0$ iterations, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when N = 200,000 iterations, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $N = N_C$ = 400,000 iterations. The other curves are the graphs of $DOK(N)$ (red) and $Chf(N)$ (green, blue, pink) in different planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, N = 200,000 iterations). The point L corresponds to ($DOK = 1$, $Chf = 0$, $N = N_C$ = 400,000 iterations). The three points J, K, L are the same as in Figure 8.

In the second cube (Figure 10), we can notice the simulation of the convergence probability $P_r(N)$ and its complementary real divergence probability $P_m(N)/i$ in terms of the iterations N for the Buffon's needle problem. The thick line in cyan is the projection of $Pc^2(N) = P_r(N) + P_m(N)/i = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point ($P_r = 0$, $P_m/i = 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $P_r(N)$ in the plane $P_r(N) = P_m(N)/i$. This curve starts at the point J ($P_r = 0$, $P_m/i = 1$, $N = 0$ iterations), reaches the point K ($P_r = 0.5$, $P_m/i = 0.5$, N = 200,000 iterations), and gets at the end to L ($P_r = 1$, $P_m/i = 0$, $N = N_C$ = 400,000 iterations). The blue curve represents $P_m(N)/i$ in the plane $P_r(N) + P_m(N)/i = 1$. Notice the importance of the point K which is the intersection of the red and blue curves at N = 200,000 iterations and when $P_r(N) = P_m(N)/i = 0.5$. The three points J, K, L are the same as in Figure 8.

In the third cube (Figure 11), we can notice the simulation of the complex random vector $Z(N)$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ as a function of the real convergence probability $P_r(N) = \mathrm{Re}(Z)$ in $\mathcal{R}$ and of its complementary imaginary divergence probability $P_m(N) = i \times \mathrm{Im}(Z)$ in $\mathcal{M}$, and this in terms of the iterations N for the Buffon's needle problem. The red curve represents $P_r(N)$ in the plane $P_m(N) = 0$ and the blue curve represents $P_m(N)$ in the plane $P_r(N) = 0$. The green curve represents the complex probability vector $Z(N) = P_r(N) + P_m(N) = \mathrm{Re}(Z) + i \times \mathrm{Im}(Z)$ in the plane $P_r(N) = iP_m(N) + 1$. The curve of $Z(N)$ starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point L ($P_r = 1$, $P_m = 0$, $N = N_C = 400{,}000$ iterations). The thick line in cyan is $P_r(0) = iP_m(0) + 1$ and it is the projection of the $Z(N)$ curve on the complex probability plane whose equation is $N = 0$ iterations. This projected line starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point ($P_r = 1$, $P_m = 0$, $N = 0$ iterations). Notice the importance of the point K corresponding to $N = 200{,}000$ iterations and when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in Figure 8.

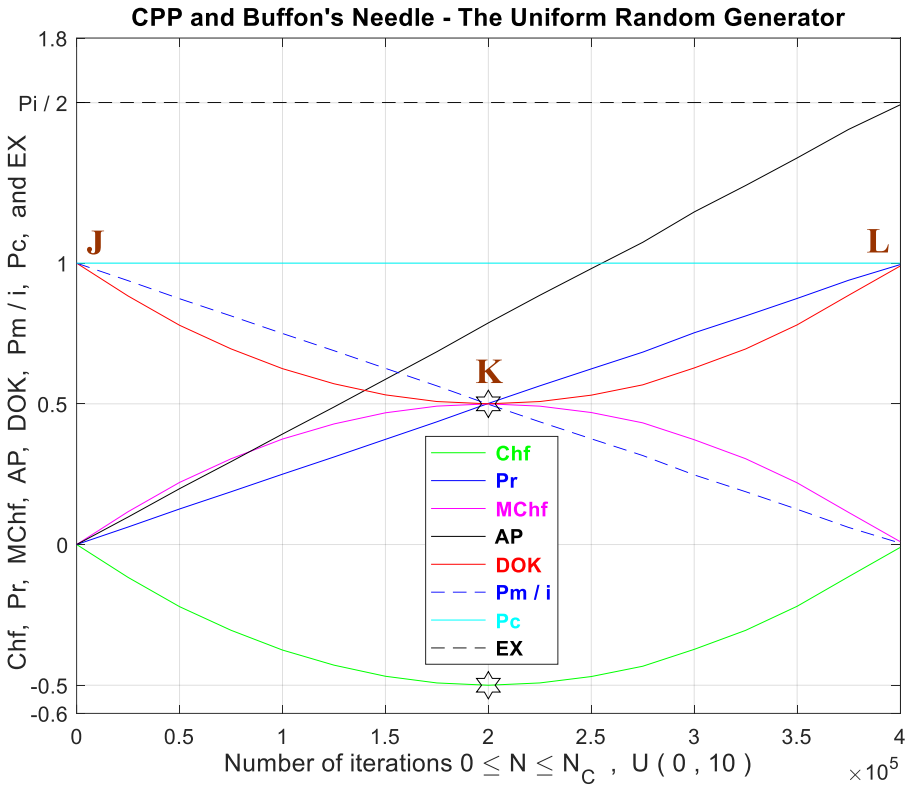


Figure 8- The *CPP* parameters and the Buffon's needle method with the Uniform random number generator.

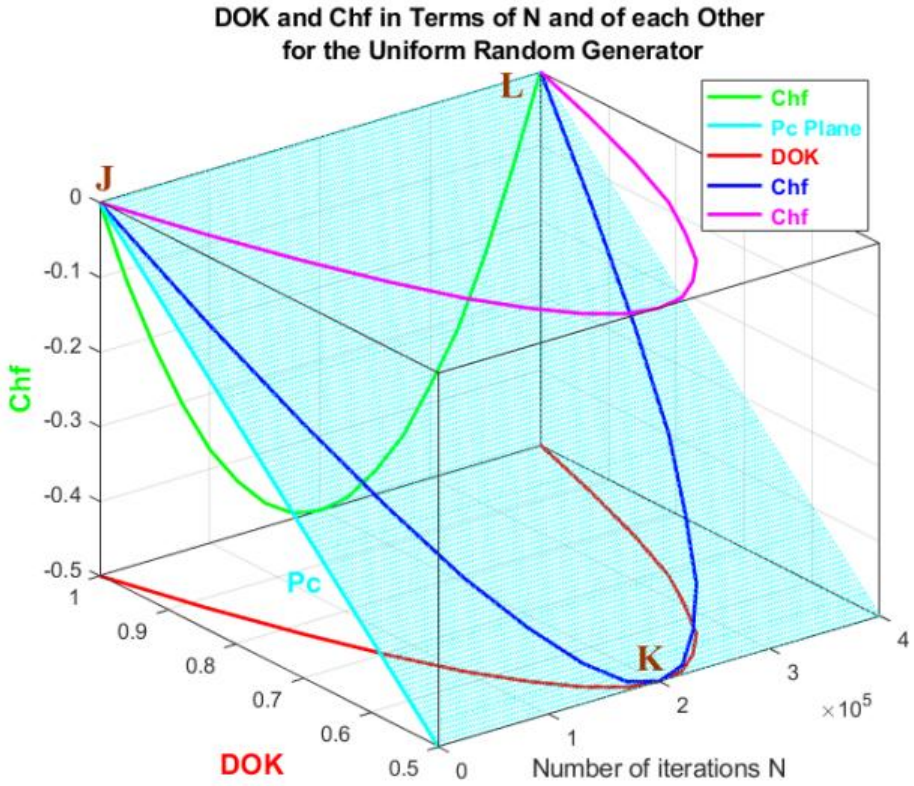


Figure 9- *DOK* and *Chf* in terms of *N* and of each other for the Buffon's needle method with the Uniform random number generator.

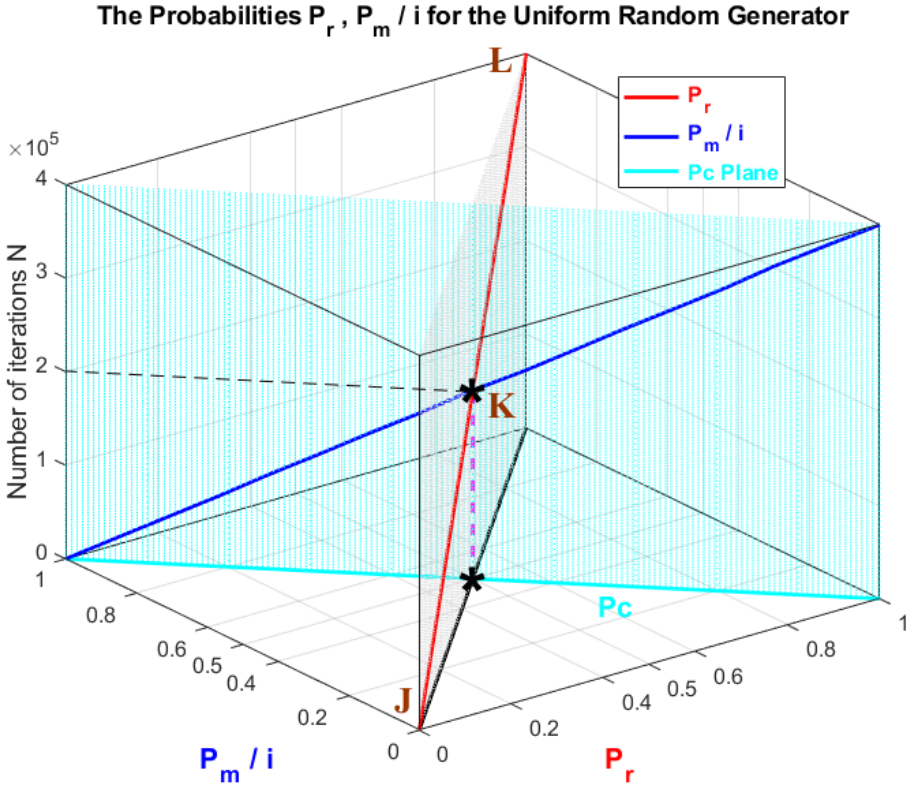


Figure 10- P_r and P_m/i in terms of *N* and of each other for the Buffon's needle method with the Uniform random number generator.

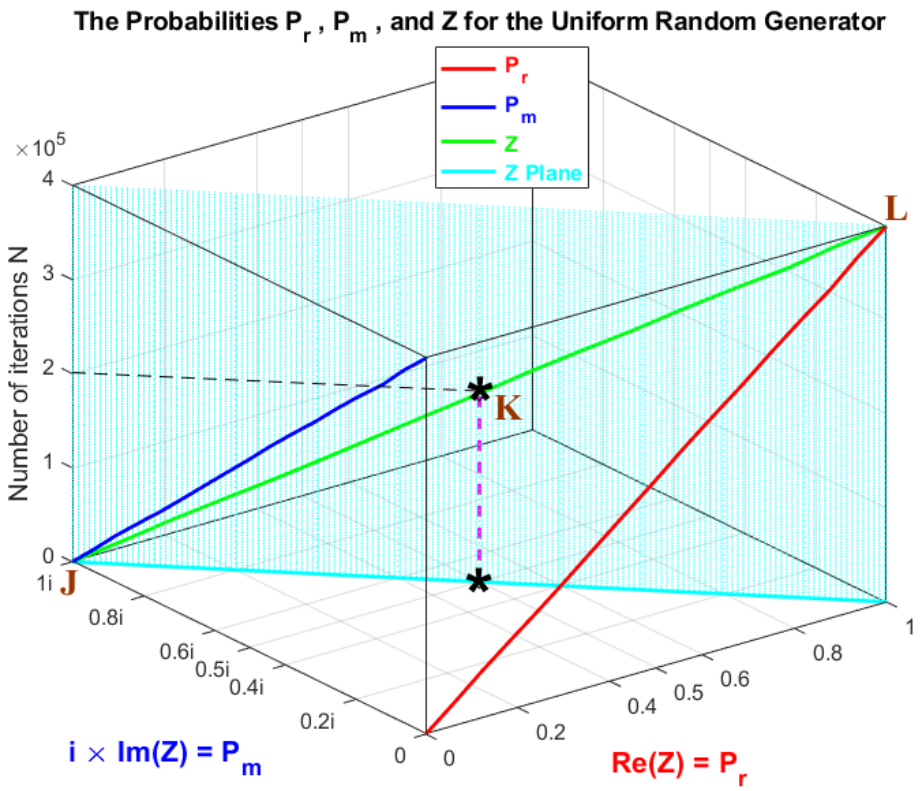


Figure 11- The Complex Probability Vector Z in terms of N for the Buffon's needle method with the Uniform random number generator.

9-2- The Gaussian and Normal Random Numbers Generator Case

We will use in the second case in the computation of π using Buffon's needle method the Gaussian and normal random numbers generator:

$$(y_a, \alpha, y_b, j, q_1, q_2) \mapsto \mathbb{N}\ (\mu = 0, \sigma = 1)$$

L which is the needle length is taken to be equal to 1.

$\Leftrightarrow AP(N) = \frac{1}{2} \times \frac{2NL}{X}$ with $1 \le N \le N_C$ after applying Buffon's needle method.

Moreover, the three figures (Figures 12-14) show the increasing convergence of Buffon's needle method and simulation to the exact result $EX = \pi/2 = 3.141592654\ldots/2 = 1.570796327\ldots$ for N = 1000, 30000, and $N_C = 400000$ iterations. Therefore, we have:

$\lim_{N \to +\infty} P_r(N) = \lim_{N \to +\infty} \left\{ 1 - \left| \frac{EX - AP(N)}{EX} \right| \right\} = 1 - \left| \frac{EX - EX}{EX} \right| = 1 - 0 = 1$ which is equal to the convergence probability of Buffon's needle method as $N \to +\infty$.

Additionally, Figure 15 illustrates clearly and visibly the relation of Buffon's needle method to the complex probability paradigm with all its parameters $(Chf, MChf, DOK, EX, AP, P_r, P_m / i, Pc)$ after applying it to this random method.

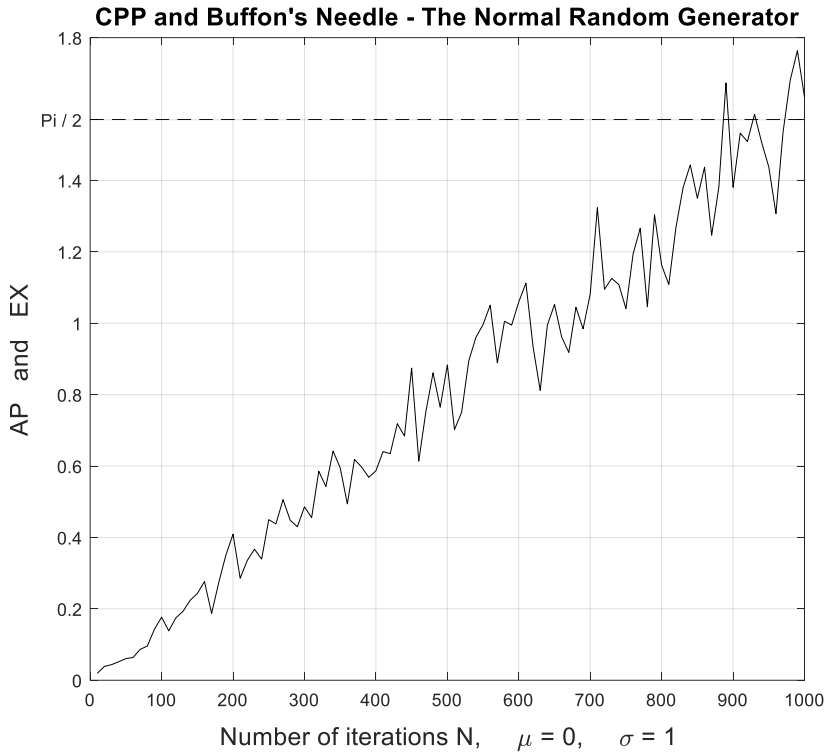


Figure 12- The increasing convergence of Buffon's needle method up to N = 1000 iterations with the Gaussian and normal random number generator.

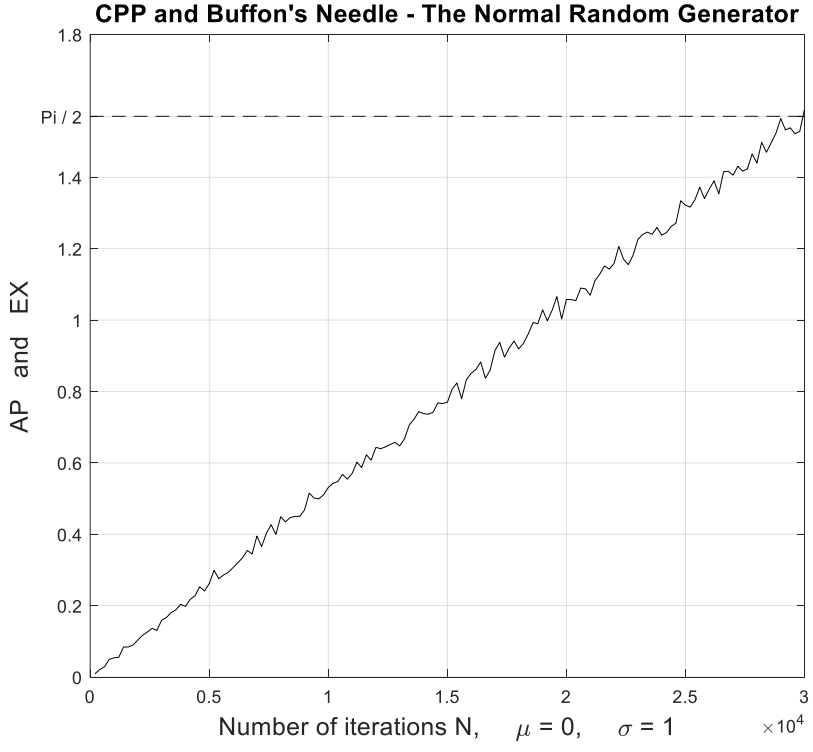


Figure 13- The increasing convergence of Buffon's needle method up to N = 30,000 iterations with the Gaussian and normal random number generator.

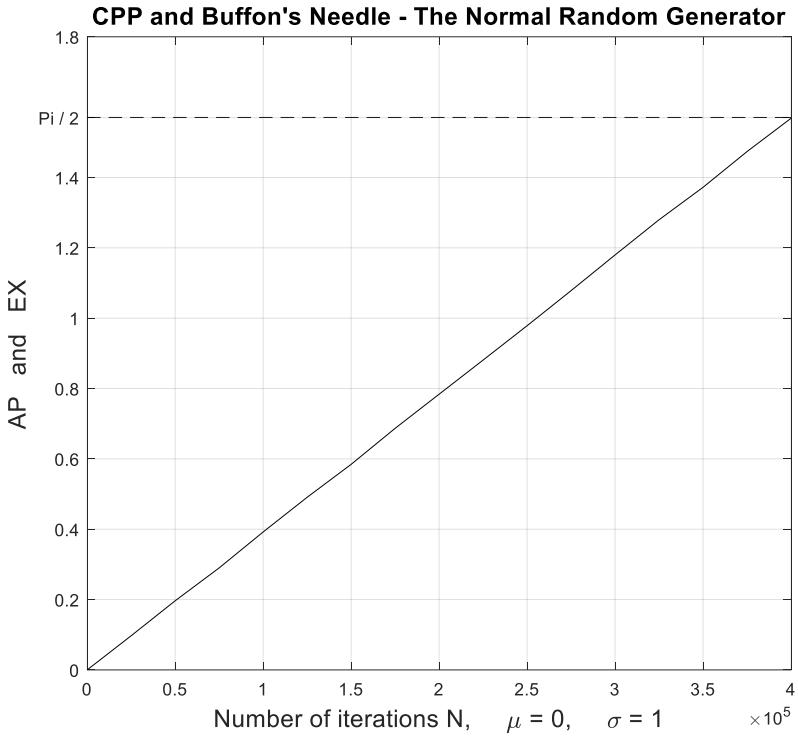


Figure 14- The increasing convergence of Buffon's needle method up to N = 400,000 iterations with the Gaussian and normal random number generator.

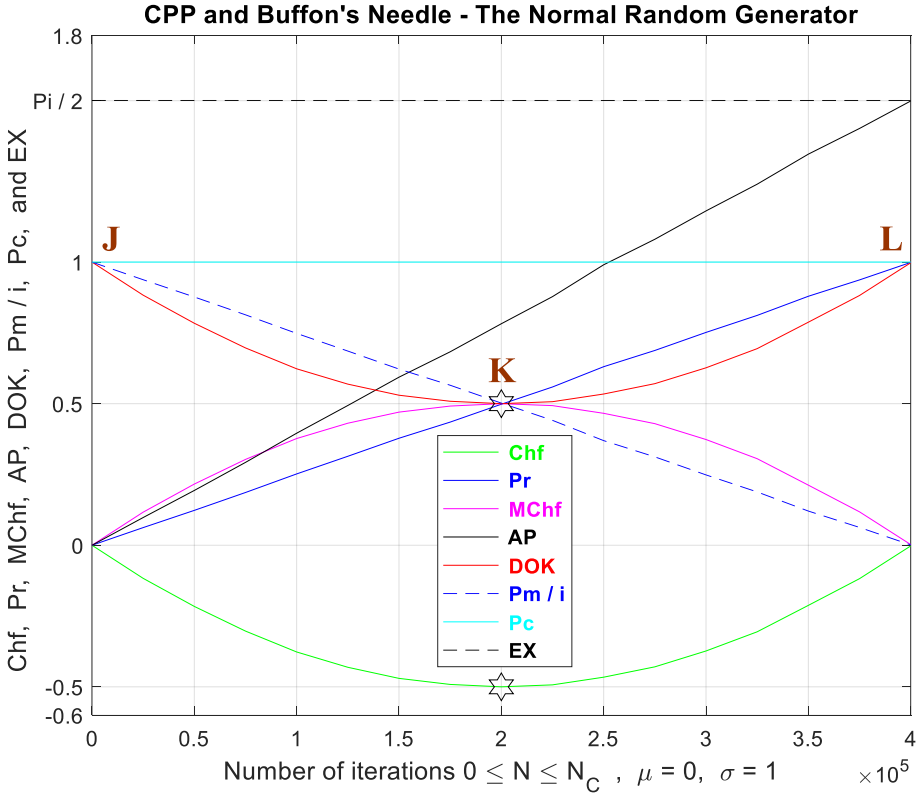


Figure 15- The *CPP* parameters and the Buffon's needle method with the Gaussian and normal random number generator.

9-3- The Poisson Random Numbers Generator Case

We will use in the third case in the computation of π using Buffon's needle method the Poisson random numbers generator:

$$(y_a, \alpha, y_b, j, q_1, q_2) \mapsto \mathbb{P} \ (\lambda = 5.68)$$

L which is the needle length is taken to be equal to 1.

$\Leftrightarrow AP(N) = \frac{1}{2} \times \frac{2NL}{X}$ with $1 \le N \le N_C$ after applying Buffon's needle method.

Moreover, the three figures (Figures 16-18) show the increasing convergence of Buffon's needle method and simulation to the exact result $EX = \pi / 2 = 3.141592654\ldots / 2 = 1.570796327\ldots$ for N = 1000, 30000, and $N_C = 400000$ iterations. Therefore, we have:

$\lim_{N \to +\infty} P_r(N) = \lim_{N \to +\infty} \left\{ 1 - \left| \frac{EX - AP(N)}{EX} \right| \right\} = 1 - \left| \frac{EX - EX}{EX} \right| = 1 - 0 = 1$ which is equal to the convergence probability of Buffon's needle method as $N \to +\infty$.

Additionally, Figure 19 illustrates clearly and visibly the relation of Buffon's needle method to the complex probability paradigm with all its parameters $(Chf, MChf, DOK, EX, AP, P_r, P_m / i, Pc)$ after applying it to this random method.

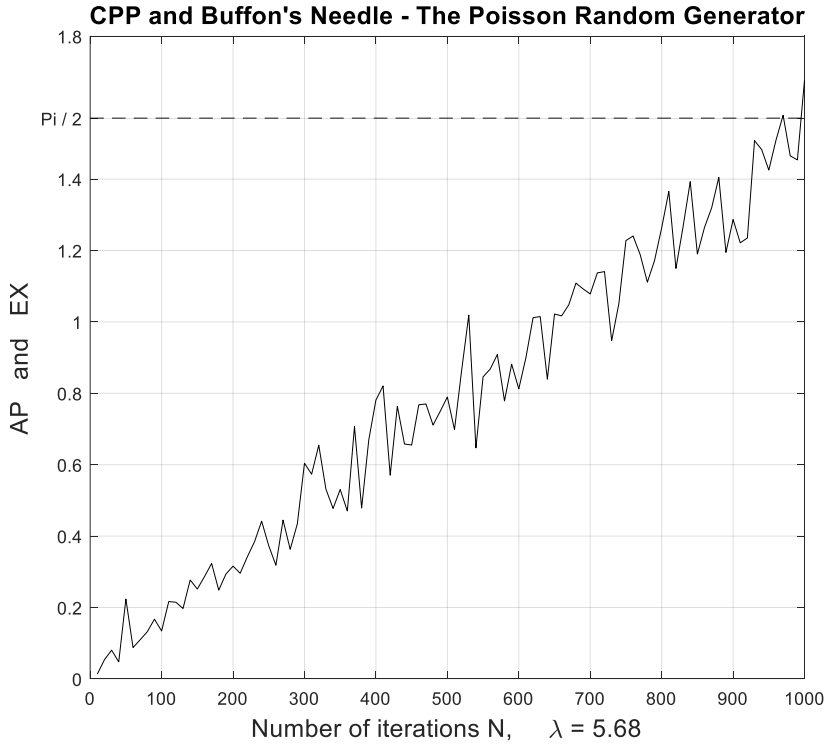


Figure 16- The increasing convergence of Buffon's needle method up to N = 1000 iterations with the Poisson random number generator.

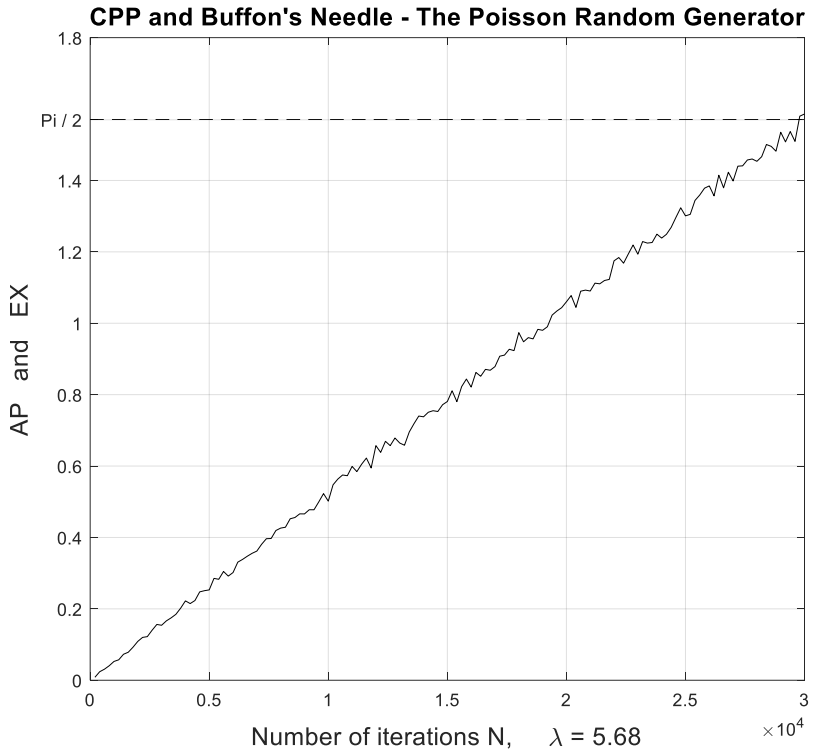


Figure 17- The increasing convergence of Buffon's needle method up to N = 30,000 iterations with the Poisson random number generator.

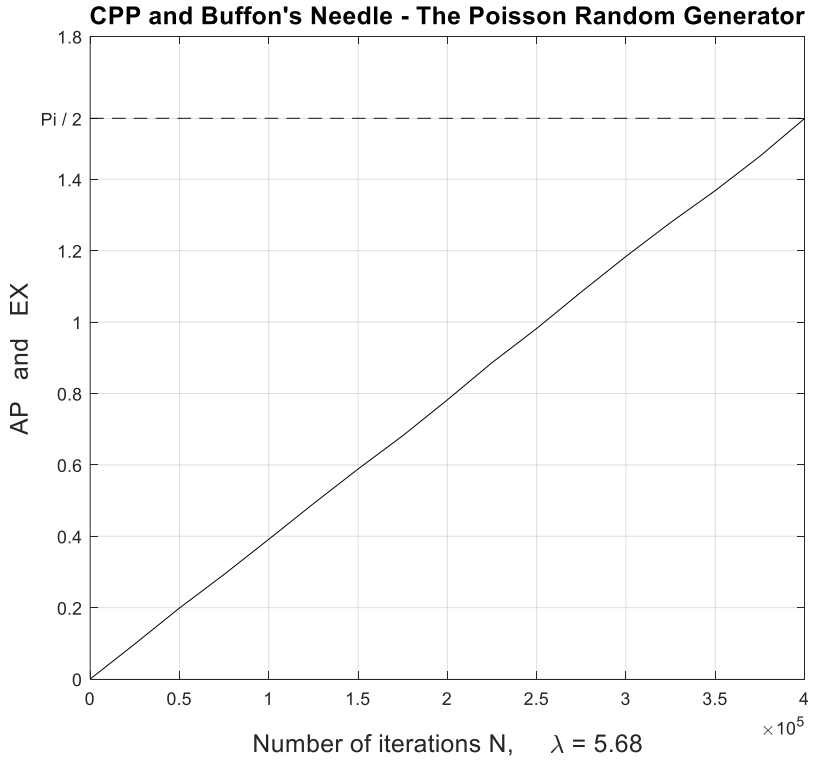


Figure 18- The increasing convergence of Buffon's needle method up to N = 400,000 iterations with the Poisson random number generator.

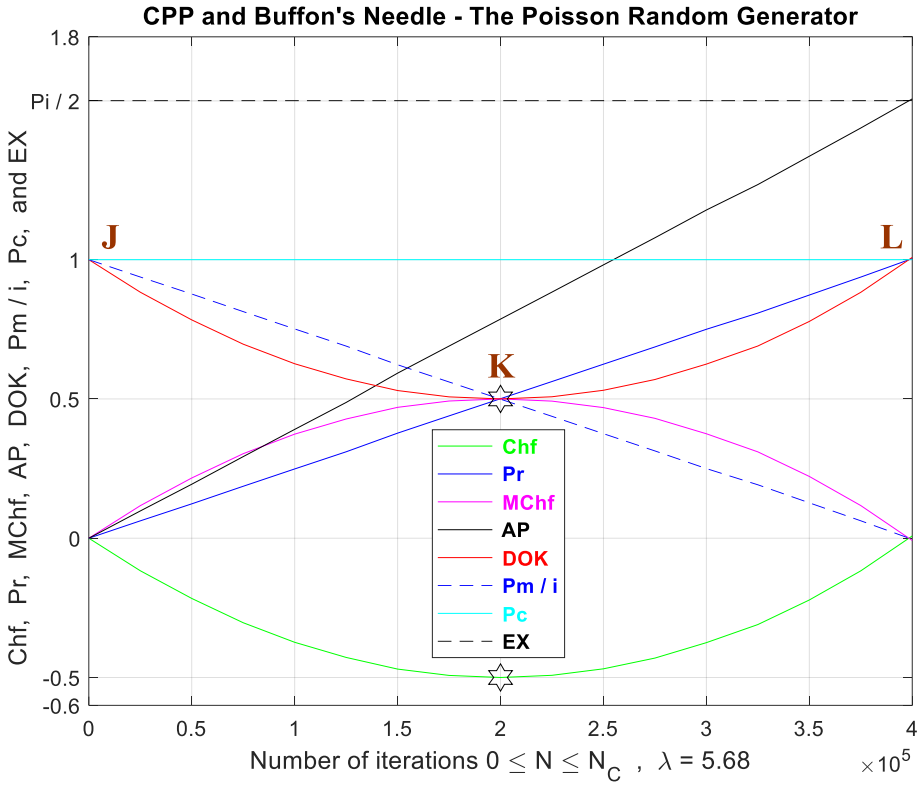


Figure 19- The *CPP* parameters and the Buffon's needle method with the Poisson random number generator.

9-4- The Weibull Random Numbers Generator Case

We will use in the fourth case in the computation of π using Buffon's needle method the Weibull random numbers generator:

$$(y_a, \alpha, y_b, j, q_1, q_2) \mapsto \mathbb{W}\ (a=1, b=2)$$

L which is the needle length is taken to be equal to 1.

$\Leftrightarrow AP(N) = \frac{1}{2} \times \frac{2NL}{X}$ with $1 \le N \le N_C$ after applying Buffon's needle method.

Moreover, the three figures (Figures 20-22) show the increasing convergence of Buffon's needle method and simulation to the exact result $EX = \pi/2 = 3.141592654\ldots/2 = 1.570796327\ldots$ for N = 1000, 30000, and $N_C = 400000$ iterations. Therefore, we have:

$\lim_{N \to +\infty} P_r(N) = \lim_{N \to +\infty} \left\{ 1 - \left| \frac{EX - AP(N)}{EX} \right| \right\} = 1 - \left| \frac{EX - EX}{EX} \right| = 1 - 0 = 1$ which is equal to the convergence probability of Buffon's needle method as $N \to +\infty$.

Additionally, Figure 23 illustrates clearly and visibly the relation of Buffon's needle method to the complex probability paradigm with all its parameters $(Chf, MChf, DOK, EX, AP, P_r, P_m/i, Pc)$ after applying it to this random method.

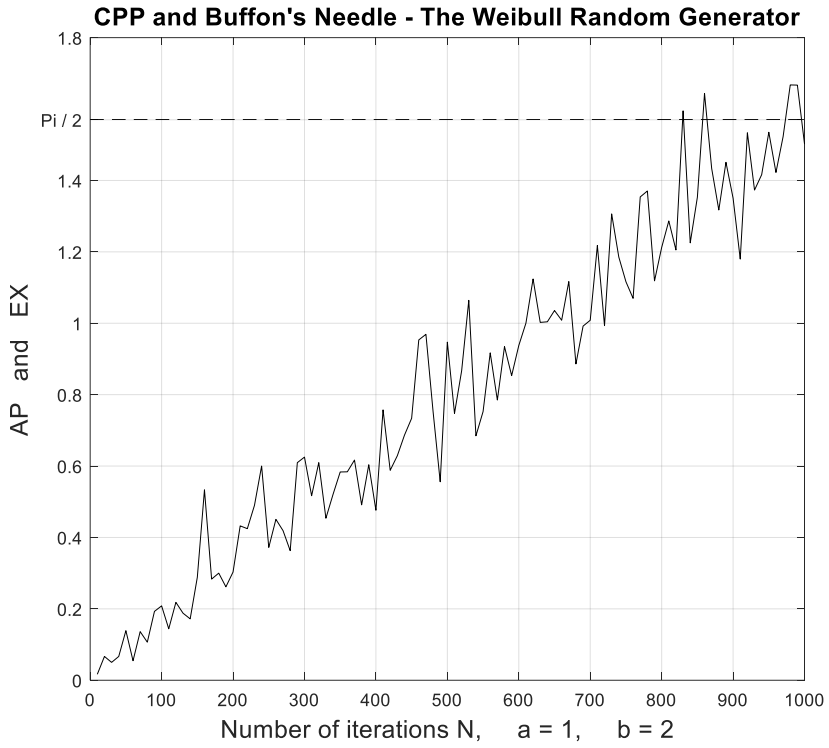


Figure 20- The increasing convergence of Buffon's needle method up to N = 1000 iterations with the Weibull random number generator.

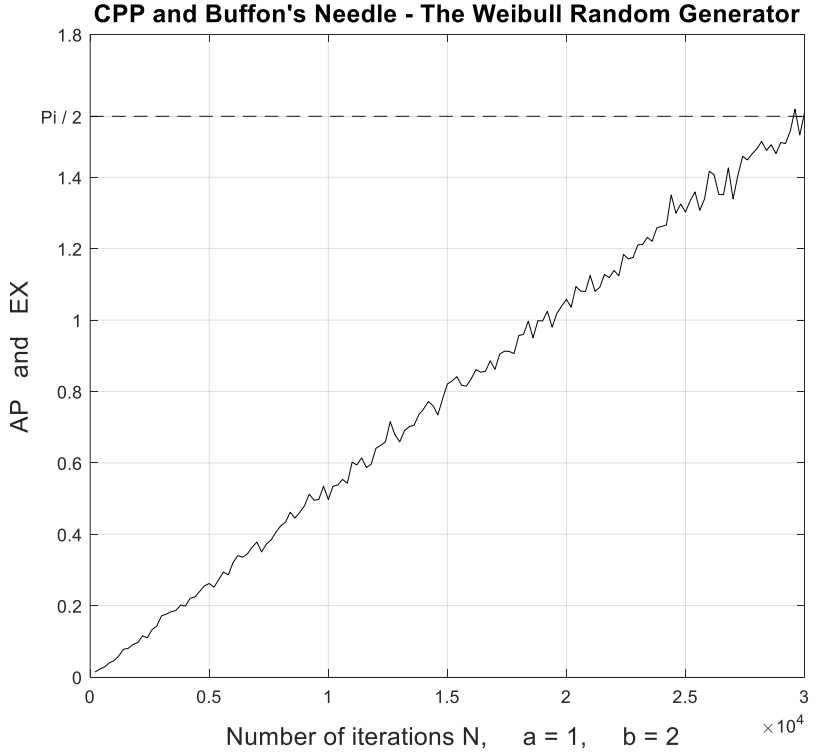


Figure 21- The increasing convergence of Buffon's needle method up to N = 30,000 iterations with the Weibull random number generator.

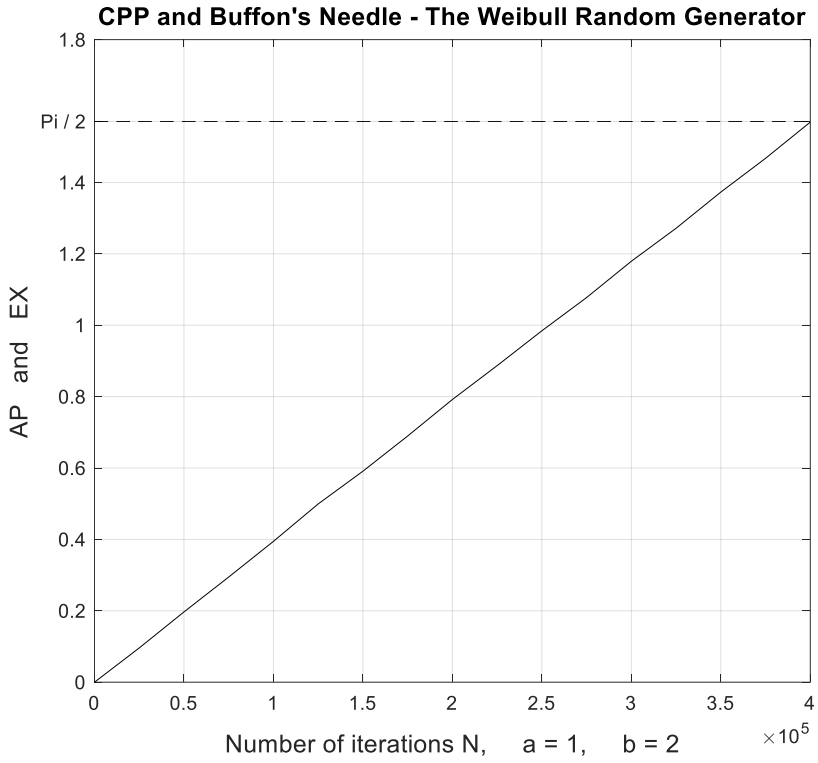


Figure 22- The increasing convergence of Buffon's needle method up to N = 400,000 iterations with the Weibull random number generator.

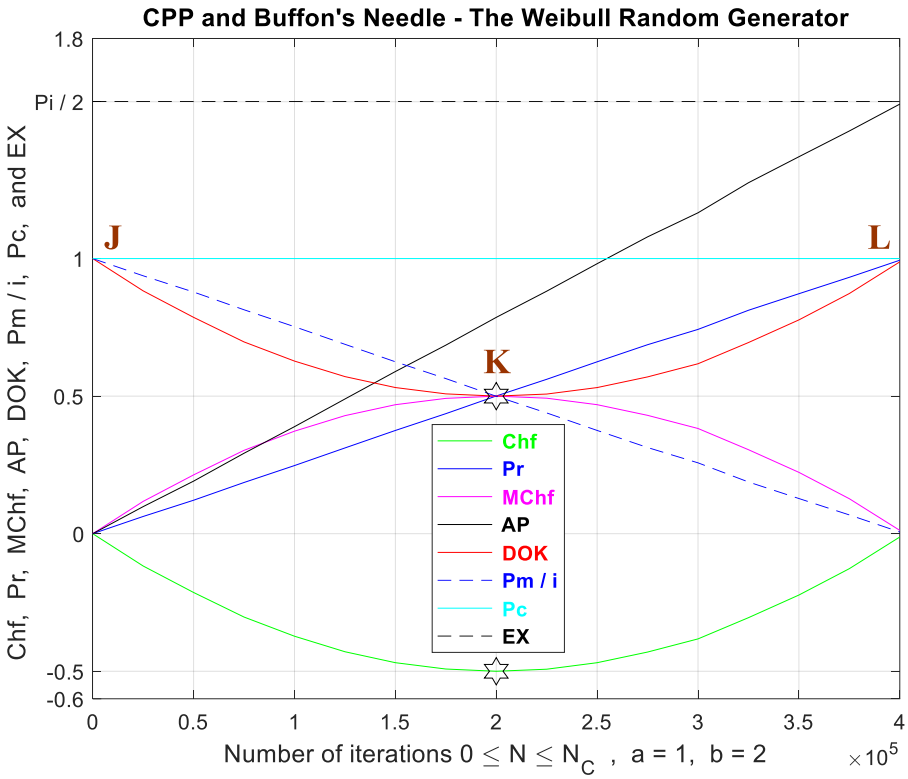


Figure 23- The *CPP* parameters and the Buffon's needle method with the Weibull random number generator.

10- Conclusion and Perspectives

In the current research chapter, the original extended Kolmogorov model of eight axioms (*EKA*) was connected and applied to the classical and random Buffon's needle numerical technique. Thus, a tight link between Buffon's needle algorithms and the novel paradigm was executed. Accordingly, the model of "Complex Probability" was more expanded beyond the scope of my earlier research studies on this subject.

Moreover, as it was proved and illustrated in the new model and in the current chapter and in all the book chapters, when the probability is 0 or 1 then the degree of our knowledge (*DOK*) is one and the chaotic factor (*Chf* and *MChf*) is 0 since the state of the system is totally known. During the random process we have: $0.5 \leq DOK < 1$, $-0.5 \leq Chf < 0$, and $0 < MChf \leq 0.5$. Notice that during this whole procedure we have always $Pc^2 = DOK - Chf = DOK + MChf = 1$, that means that the phenomenon which seems to be random and stochastic in $\mathcal{R}$ is now deterministic and certain in $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after adding to $\mathcal{R}$ the contributions of $\mathcal{M}$ and hence after subtracting the chaotic factor from the degree of our knowledge. Additionally, the probabilities of convergence and divergence of the random Buffon's needle procedure that correspond to each iteration cycle *N* have been determined in the three sets of probabilities which are $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$ by P_r, P_m, and *Pc* respectively. Subsequently, at each instance of *N*, the novel Buffon's needle technique and *CPP* parameters EX, AP, P_r, P_m, P_m / i, *DOK*, *Chf*, *MChf*, *Pc*, and *Z* are perfectly and surely predicted in the set of complex probabilities $\mathcal{C}$ with *Pc* kept as equal to 1 continuously and forever.

Furthermore, using all these shown simulations and obtained graphs all over the entire research chapter, we can visualize and quantify both the certain knowledge (expressed by *DOK* and *Pc*) and the system chaos and stochastic influences and effects (expressed by *Chf* and *MChf*) of Buffon's needle algorithms. This is definitely very wonderful, fruitful, and fascinating and demonstrates once again the advantages of extending the five axioms of probability of Kolmogorov and thus the benefits and novelty of this original theory in applied mathematics and prognostics that can be called verily:

"The Complex Probability Paradigm".

Moreover, it is important to state here that four essential and very well-known random numbers generators were taken into consideration in the current chapter which are the uniform, Gaussian, Poisson and Weibull random numbers generators, knowing that the original *CPP* model can be applied to any generator of random numbers that exist in literature. This will lead certainly to analogous results and conclusions and will confirm without any doubt the success of my innovative theory.

As a prospective and future work and concerning some applications to pure and applied mathematics, it is planned to more develop the novel proposed mathematical prognostic paradigm and to apply it to a wide set of random and stochastic systems in various fields of science and disciplines of knowledge.

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this research work.

CHAPTER FOUR

THE PARADIGM OF COMPLEX PROBABILITY AND THE NEUTRON SHIELDING PROBLEM

"It is not certain that everything is uncertain."
Blaise Pascal, Pascal's Pensées.

"A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."
Godfrey Harold Hardy, A Mathematician's Apology.

"I would prefer an intelligent hell to a stupid paradise."
Blaise Pascal.

"A mathematical proof should resemble a simple and clear-cut constellation, not a scattered cluster in the Milky Way."
Godfrey Harold Hardy, A Mathematician's Apology.

Abstract: The concept of mathematical probability was established in 1933 by *Andrey Nikolaevich Kolmogorov* by defining a system of five axioms. This system can be enhanced to encompass the imaginary numbers set after the addition of three novel axioms. As a result, any random experiment can be executed in the complex probabilities set $\mathcal{C}$ which is the sum of the real probabilities set $\mathcal{R}$ and the imaginary probabilities set $\mathcal{M}$. We aim here to incorporate supplementary imaginary dimensions to the random experiment occurring in the "real" laboratory in $\mathcal{R}$ and therefore to compute all the probabilities in the sets $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$. Accordingly, the probability in the whole set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is constantly equivalent to one independently of the distribution of the input random variable in $\mathcal{R}$, and subsequently the output of the stochastic experiment in $\mathcal{R}$ can be determined absolutely in $\mathcal{C}$. This is the consequence of the fact that the probability in $\mathcal{C}$ is computed after the subtraction of the chaotic factor from the degree of our knowledge of the nondeterministic experiment. We will apply this innovative paradigm to the well-known neutron shielding problem and to its random algorithms and procedures in a novel way.

Keywords: Degree of our knowledge, Chaotic factor, Complex probability set, Probability norm, Complex random vector, Convergence probability, Divergence probability, Simulation.

NOMENCLATURE

$\mathcal{R}$ = the events real set
$\mathcal{M}$ = the events imaginary set
$\mathcal{C}$ = the events complex set

i = the imaginary number with $i^2 = -1$ or $i = \sqrt{-1}$
EKA = Extended Kolmogorov's Axioms
CPP = Complex Probability Paradigm
P_{rob} = any event probability
P_r = the probability in the real set $\mathcal{R}$ = convergence probability in $\mathcal{R}$
P_m = the probability in the complementary imaginary set $\mathcal{M}$ that corresponds to the real probability set in $\mathcal{R}$ = divergence probability in $\mathcal{M}$
Pc = the event probability in $\mathcal{R}$ with its associated event in $\mathcal{M}$ = probability in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$
EX = the random experiment exact result
AP = the random experiment approximate result
Z = complex probability number = complex random vector = sum of P_r and P_m
DOK = $|Z|^2$ = the degree of our knowledge of the stochastic experiment or system, it is the square of the norm of *Z*
Chf = the chaotic factor of *Z*
MChf = the magnitude of the chaotic factor of *Z*
N = the number of iterations cycles = number of random vectors
N_C = the number of iterations cycles till the convergence of neutron shielding problem to *EX* = the number of random vectors till convergence.

1- Introduction [1-90]

Calculating probabilities is the crucial task of classical probability theory. Adding supplementary dimensions to nondeterministic experiments will yield a deterministic expression of the theory of probability. This is the novel and original idea at the foundations of my complex probability paradigm. Aa a matter of fact, probability theory is a stochastic system of axioms in its essence; that means that the phenomena outputs are due to randomness and chance. By adding novel imaginary dimensions to the nondeterministic phenomenon happening in the set $\mathcal{R}$ will lead to a deterministic phenomenon and thus a stochastic experiment will have a certain output in the complex probability set $\mathcal{C}$. If the chaotic experiment becomes completely predictable then we will be fully capable to predict the output of random events that arise in the real world in all stochastic processes. Accordingly, the task that has been achieved here was to extend the random real probabilities set $\mathcal{R}$ to the deterministic complex probabilities set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ and this by incorporating the contributions of the set $\mathcal{M}$ which is the complementary imaginary set of probabilities to the set $\mathcal{R}$. Consequently, since this extension reveals to be successful, then an innovative paradigm of stochastic sciences and prognostic was put forward in which all nondeterministic phenomena in $\mathcal{R}$ was expressed deterministically in $\mathcal{C}$. I coined this novel model by the term "The Complex Probability Paradigm" that was initiated and established in my earlier research works.

2- The Purpose and the Advantages of the Current Chapter [37-90]

The advantages and the purpose of the present chapter are to:

1- Extend the theory of classical probability to cover the complex numbers set, hence to connect the probability theory to the field of complex variables and analysis. This task was started and elaborated in my earlier papers.

2- Apply the novel probability axioms and paradigm to the neutron shielding problem.

3- Show that all nondeterministic phenomena can be expressed deterministically in the complex probabilities set which is $\mathcal{C}$.

4- Compute and quantify both the degree of our knowledge and the chaotic factor of the neutron shielding problem.

5- Represent and show the graphs of the functions and parameters of the innovative paradigm related to the neutron shielding algorithm.

6- Demonstrate that the classical concept of probability is permanently equal to one in the set of complex probabilities; hence, no chaos, no randomness, no ignorance, no uncertainty, no unpredictability, no nondeterminism, and no disorder exist in:

$$\mathcal{C}\ (\text{complex set}) = \mathcal{R}\ (\text{real set}) + \mathcal{M}\ (\text{imaginary set}).$$

7- Pave the way to implement this inventive model to other topics in prognostics and to the field of stochastic processes. These will be the goals of my future research works.

Concerning some applications of the novel established paradigm and as a future work, it can be applied to any nondeterministic phenomena using the neutron shielding algorithm in any random case.

Moreover, compared with existing literature, the major contribution of the current research chapter is to apply the innovative paradigm of complex probability to the concept and technique of the probabilistic neutron shielding simulation and algorithms. The next figure displays the major aims and purposes of the Complex Probability Paradigm (*CPP*) (Figure 1).

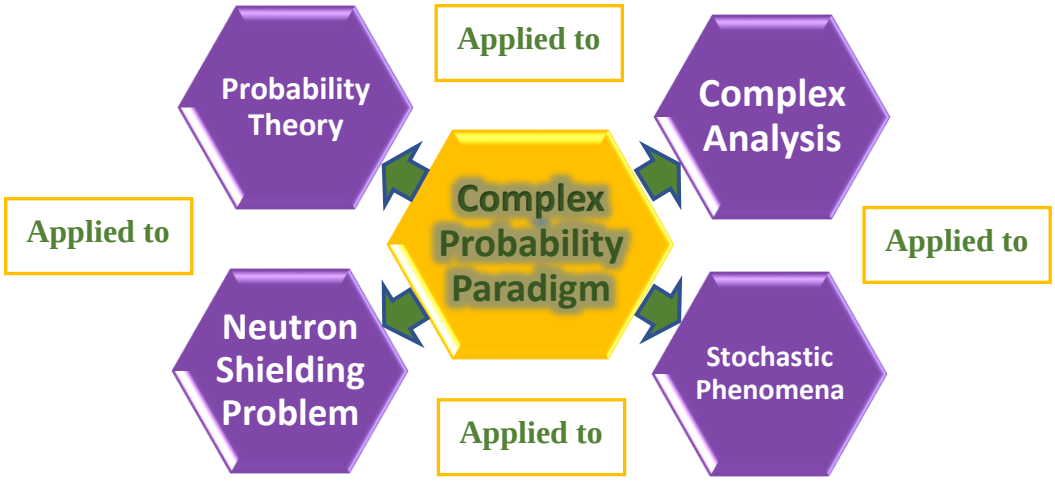


Figure 1- The diagram of the major aims of the Complex Probability Paradigm and the neutron shielding problem.

3- The Complex Probability Paradigm [37-141]

3-1- The Original Andrey Nikolaevich Kolmogorov System of Axioms

The simplicity of Kolmogorov's system of axioms may be surprising. Let E be a collection of elements $\{E_1, E_2, \ldots\}$ called elementary events and let F be a set of subsets of E called random events. The five axioms for a finite set E are:

Axiom 1: F is a field of sets.
Axiom 2: F contains the set E.
Axiom 3: A non-negative real number $P_{rob}(A)$, called the probability of A, is assigned to each set A in F. We have always $0 \le P_{rob}(A) \le 1$.
Axiom 4: $P_{rob}(E)$ equals 1.
Axiom 5: If A and B have no elements in common, the number assigned to their union is:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B)$$

hence, we say that A and B are disjoint; otherwise, we have:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B) - P_{rob}(A \cap B)$$

And we say also that: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B/A) = P_{rob}(B) \times P_{rob}(A/B)$ which is the conditional probability. If both A and B are independent then: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B)$.

Moreover, we can generalize and say that for N disjoint (mutually exclusive) events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following additivity rule:

$$P_{rob}\left(\bigcup_{j=1}^{N} A_j\right)=\sum_{j=1}^{N} P_{rob}\left(A_j\right)$$

And we say also that for N independent events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \le j \le N$), we have the following product rule:

$$P_{rob}\left(\bigcap_{j=1}^{N} A_j\right)=\prod_{j=1}^{N} P_{rob}\left(A_j\right)$$

3-2- Adding the Imaginary Part $\mathcal{M}$

Now, we can add to this system of axioms an imaginary part such that:

Axiom 6: Let $P_m = i \times (1 - P_r)$ be the probability of an associated complementary event in $\mathcal{M}$ (the imaginary part) to the event A in $\mathcal{R}$ (the real part). It follows that $P_r + P_m / i = 1$ where i is the imaginary number with $i = \sqrt{-1}$ or $i^2 = -1$.

Axiom 7: We construct the complex number or vector $Z = P_r + P_m = P_r + i(1 - P_r)$ having a norm $|Z|$ such that:

$$|Z|^2 = P_r^2 + (P_m / i)^2 .$$

Axiom 8: Let *Pc* denote the probability of an event in the complex probability universe $\mathcal{C}$ where $\mathcal{C} = \mathcal{R} + \mathcal{M}$. We say that *Pc* is the probability of an event A in $\mathcal{R}$ with its associated event in $\mathcal{M}$ such that:

$$Pc^2 = (P_r + P_m / i)^2 = |Z|^2 - 2iP_rP_m \text{ and is always equal to 1.}$$

We can see that by taking into consideration the set of imaginary probabilities we added three new and original axioms and consequently the system of axioms defined by Kolmogorov was hence expanded to encompass the set of imaginary numbers.

3-3- A Brief Interpretation of the Novel Paradigm

To summarize the novel paradigm, we state that in the real probability universe $\mathcal{R}$ our degree of our certain knowledge is undesirably imperfect and hence unsatisfactory, thus we extend our analysis to the set of complex numbers $\mathcal{C}$ which incorporates the contributions of both the set of real probabilities which is $\mathcal{R}$ and the complementary set of imaginary probabilities which is $\mathcal{M}$. Afterward, this will yield an absolute and perfect degree of our knowledge in the probability universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ because $Pc = 1$ constantly. As a matter of fact, the work in the complex universe $\mathcal{C}$ gives way to a sure prediction of any stochastic experiment, because in $\mathcal{C}$ we remove and subtract from the computed degree of our knowledge the measured chaotic factor. This will generate a probability in the universe $\mathcal{C}$ equal to 1 ($Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$). Many illustrations taking into consideration numerous continuous and discrete probability distributions in my previous research papers confirm this hypothesis and innovative paradigm. The Extended Kolmogorov Axioms (*EKA* for short) or the Complex Probability Paradigm (*CPP* for short) can be shown and summarized in the next illustration (Figure 2):

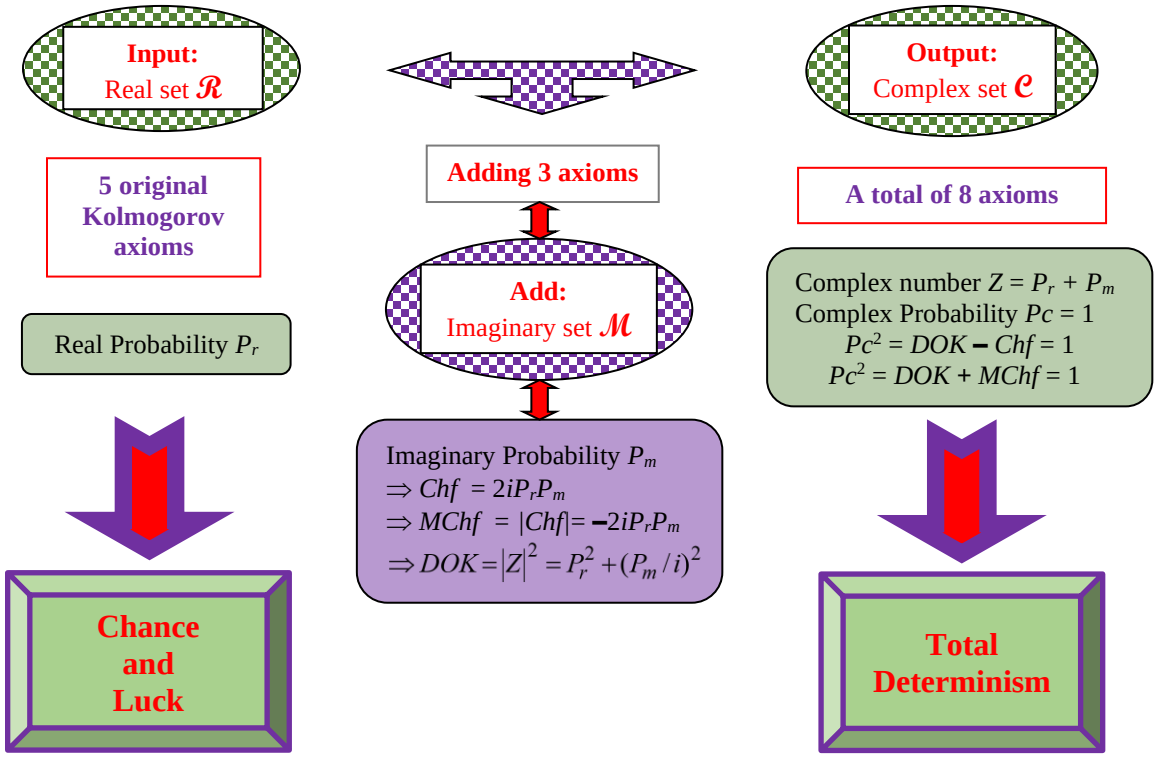


Figure 2- The *EKA* or the *CPP* diagram

4- The Neutron Shielding Method Problem and Solution

The Neutron Shielding Problem: The work done in this paper concerns the neutron shielding. We take a simple model of neutrons penetrating a lead wall. It is assumed that each neutron enters the lead wall at a right angle to the wall and travels a unit distance. Then it collides with a lead atom and rebounds in a random direction. Again, it travels a unit distance before colliding with another lead atom. It rebounds in a random direction and so on. Assume that after 11 collisions, all the neutron's energy is spent. Assume also that the lead wall is 5 units thick in the *X* direction and for all practical purposes infinitely thick in the *Y* direction. The question is: *what percentage of neutrons can be expected to emerge from the other side of the lead wall*?

The Neutron Shielding Solution: Let *x* be the distance measured from the initial surface where the neutron enters. From trigonometry, we recall that in a right triangle with hypotenuse 1, one side is $\cos\theta$. Also note that $\cos\theta \le 0$ when $\pi/2 \le \theta \le \pi$. The first collision occurs at a point where $x = 1$. The second occurs at a point where $x = 1 + \cos\theta_1$. The third collision occurs at a point where $x = 1 + \cos\theta_1 + \cos\theta_2$, and so on. If $x \ge 5$, the neutron has exited. If $x < 5$ for all eight collisions, the wall has shielded the area from that particular neutron. The figures below justify our mathematical analysis (Figures 3 and 4):

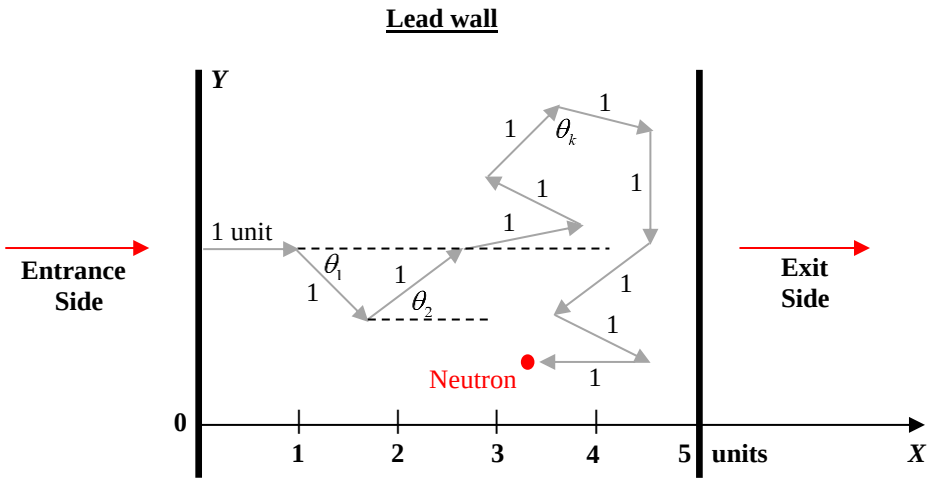


Figure 3- The neutrons penetrating a lead wall

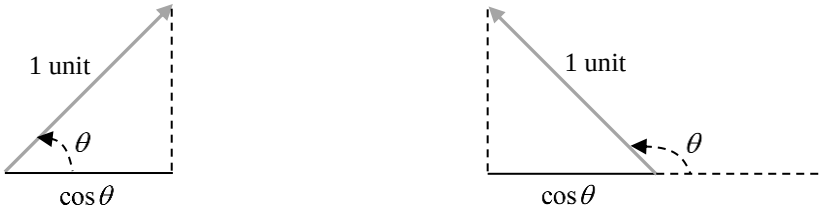


Figure 4- The neutrons travelling a unit distance

After running the program, we can say that: Exact Result = $EX \cong 5.1875\%$. So, 5.187% of the neutrons can be expected to emerge from the lead wall and which is the correct answer up to four significant digits after truncation, and that was taken from the programs' simulations. Inside the computer program, the variable *counter* counts the number of times the neutrons emerge from the other side of the lead wall, and the variable N = number of iterations such that: $0 \le N \le N_C$; therefore:

$$\text{Approximate Result} = 100 \times \frac{counter}{N}\% = \text{the variable } AP \text{ in the whole work.}$$

5- The Neutron Shielding Method and the Complex Probability Paradigm Parameters

5-1- The Convergence and Divergence Probabilities

Let EX be the experiment exact result like of a multidimensional or a simple integral that is not possible always to determine by ordinary methods or calculus or numerical deterministic methods. And let AP be the experiment approximate result and therefore let it be the value of these random experiments found by Monte Carlo Techniques like the neutron shielding problem.

The absolute error in the numerical analysis method is: $\text{Abs. Error} = |EX - AP|$

The relative error in the numerical method is: $\text{Rel. Error} = \left|\frac{\text{Abs. Error}}{EX}\right| = \left|\frac{EX - AP}{EX}\right| = \left|1 - \frac{AP}{EX}\right|$

In addition, the percent relative error is = $100\% \times \left|\frac{EX - AP}{EX}\right|$ and is always between 0% and 100%.

Therefore, the relative error is always between 0 and 1. Hence:

$$0 \le \left|\frac{EX - AP}{EX}\right| \le 1 \Leftrightarrow \begin{cases} 0 \le \left(\frac{EX - AP}{EX}\right) \le 1 & \text{if } AP \le EX \\ 0 \le -\left(\frac{EX - AP}{EX}\right) \le 1 & \text{if } AP \ge EX \end{cases} \Leftrightarrow \begin{cases} 0 \le AP \le EX \\ EX \le AP \le 2EX \end{cases}$$

Moreover, we define the real probability by:

$$P_r = 1 - \left|\frac{EX - AP}{EX}\right| = 1 - \left|1 - \frac{AP}{EX}\right| = \begin{cases} 1 - \left(1 - \frac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ 1 + \left(1 - \frac{AP}{EX}\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} \frac{AP}{EX} & \text{if } 0 \le AP \le EX \\ 2 - \frac{AP}{EX} & \text{if } EX \le AP \le 2EX \end{cases}$$

= 1 – the relative error in the numerical method
= Probability and degree of the numerical method convergence in $\mathbb{R}$
= Probability and ratio of the approximate result to the exact result.
And therefore:

$$P_m = i(1 - P_r) = i\left\{1 - \left[1 - \left|\frac{EX - AP}{EX}\right|\right]\right\} = i\left\{1 - \left[1 - \left|1 - \frac{AP}{EX}\right|\right]\right\} = i\left|1 - \frac{AP}{EX}\right|$$

$$= \begin{cases} i\left(1 - \frac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ -i\left(1 - \frac{AP}{EX}\right) & \text{if } EX \le AP \le 2EX \end{cases} = \begin{cases} i\left(1 - \frac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ i\left(\frac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

= The relative error in the numerical method in $\mathbb{M}$
= Probability and degree of the numerical method divergence in the imaginary probability set $\mathbb{M}$ since it is the imaginary complement of P_r.

Consequently,

$$P_m / i = 1 - P_r = \left| 1 - \frac{AP}{EX} \right| = \begin{cases} 1 - \dfrac{AP}{EX} & \text{if } 0 \le AP \le EX \\ \dfrac{AP}{EX} - 1 & \text{if } EX \le AP \le 2EX \end{cases}$$

= The relative error in the numerical method in $\mathcal{R}$

= Probability and degree of the numerical method divergence in $\mathcal{R}$ since it is the real complement of P_r.

In the case where $0 \le AP \le EX$ we have $0 \le \frac{AP}{EX} \le 1 \Rightarrow 0 \le P_r \le 1$ and we deduce also that $0 \le \left(1 - \frac{AP}{EX} \right) \le 1 \Rightarrow 0 \le P_m / i \le 1$ and $\Rightarrow 0 \le P_m \le i$

And in the case where $EX \le AP \le 2EX \Rightarrow 1 \le \frac{AP}{EX} \le 2 \Rightarrow 0 \le \left(2 - \frac{AP}{EX} \right) \le 1 \Rightarrow 0 \le P_r \le 1$ and we deduce also that $0 \le \left(\frac{AP}{EX} - 1 \right) \le 1 \Rightarrow 0 \le P_m / i \le 1$ and $\Rightarrow 0 \le P_m \le i$

Therefore, if $AP = 0$ or $AP = 2EX$ that means before the beginning of the numerical method and the simulation, then:

$$P_r = P_{rob}(\text{convergence}) \text{ in } \mathcal{R} = 0$$
$$P_m = P_{rob}(\text{divergence}) \text{ in } \mathcal{M} = i$$
$$P_m / i = P_{rob}(\text{divergence}) \text{ in } \mathcal{R} = 1$$

And if $AP = EX$ that means at the end of the simulation and the numerical method then:

$$P_r = P_{rob}(\text{convergence}) \text{ in } \mathcal{R} = 1$$
$$P_m = P_{rob}(\text{divergence}) \text{ in } \mathcal{M} = 0$$
$$P_m / i = P_{rob}(\text{divergence}) \text{ in } \mathcal{R} = 0$$

5-2- The Complex Random and Random Vector Z in $\mathcal{C} = \mathcal{R} + \mathcal{M}$

We have $Z = P_r + P_m = \begin{cases} \dfrac{AP}{EX} + i\left(1 - \dfrac{AP}{EX} \right) & \text{if } 0 \le AP \le EX \\ \left(2 - \dfrac{AP}{EX} \right) + i\left(\dfrac{AP}{EX} - 1 \right) & \text{if } EX \le AP \le 2EX \end{cases}$

where $\mathrm{Re}(Z) = P_r = \begin{cases} \dfrac{AP}{EX} & \text{if } 0 \le AP \le EX \\ 2 - \dfrac{AP}{EX} & \text{if } EX \le AP \le 2EX \end{cases}$ = the real part of Z

and $\mathrm{Im}(Z) = P_m / i = \begin{cases} 1 - \dfrac{AP}{EX} & \text{if } 0 \le AP \le EX \\ \dfrac{AP}{EX} - 1 & \text{if } EX \le AP \le 2EX \end{cases}$ = the imaginary part of *Z*.

That means that the complex random vector *Z* is the sum in $\mathcal{C}$ of the convergence real probability in $\mathcal{R}$ and of the divergence imaginary probability in $\mathcal{M}$.

If $AP = 0$ or $AP = 2EX$ (before beginning of the simulation) then $P_r = \dfrac{AP}{EX} = \dfrac{0}{EX} = 0$ or $P_r = 2 - \dfrac{AP}{EX} = 2 - \dfrac{2EX}{EX} = 2 - 2 = 0$ and $P_m = i\left(1 - \dfrac{AP}{EX}\right) = i\left(1 - \dfrac{0}{EX}\right) = i(1-0) = i$ or $P_m = i\left(\dfrac{AP}{EX} - 1\right) = i\left(\dfrac{2EX}{EX} - 1\right) = i(2-1) = i$ therefore $Z = 0 + i = i$.

If $AP = \dfrac{EX}{2}$ or $AP = \dfrac{3EX}{2}$ (at the middle of the simulation) then:

$$P_r = \begin{cases} \dfrac{AP}{EX} & \text{if } 0 \le AP \le EX \\ 2 - \dfrac{AP}{EX} & \text{if } EX \le AP \le 2EX \end{cases} = \begin{cases} \dfrac{EX}{2EX} = 0.5 & \text{if } 0 \le AP \le EX \\ 2 - \dfrac{3EX}{2EX} = 0.5 & \text{if } EX \le AP \le 2EX \end{cases}$$

$\Leftrightarrow P_r = 0.5$

$$\text{and } P_m = \begin{cases} i\left(1 - \dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ i\left(\dfrac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases} = \begin{cases} i\left(1 - \dfrac{EX}{2EX}\right) = 0.5i & \text{if } 0 \le AP \le EX \\ i\left(\dfrac{3EX}{2EX} - 1\right) = 0.5i & \text{if } EX \le AP \le 2EX \end{cases}$$

$\Leftrightarrow P_m = 0.5i$

therefore $Z = 0.5 + 0.5i$.

If $AP = EX$ (at the simulation end) then:

$$P_r = \begin{cases} \dfrac{AP}{EX} = \dfrac{EX}{EX} = 1 & \text{if } 0 \le AP \le EX \\ 2 - \dfrac{AP}{EX} = 2 - \dfrac{EX}{EX} = 2 - 1 = 1 & \text{if } EX \le AP \le 2EX \end{cases} \Leftrightarrow P_r = 1$$

And

$$P_m=\begin{cases} i\left(1-\dfrac{AP}{EX}\right) & \text{if } 0\le AP\le EX \\ i\left(\dfrac{AP}{EX}-1\right) & \text{if } EX\le AP\le 2EX \end{cases} =\begin{cases} i\left(1-\dfrac{EX}{EX}\right) & \text{if } 0\le AP\le EX \\ i\left(\dfrac{EX}{EX}-1\right) & \text{if } EX\le AP\le 2EX \end{cases}$$

$$=\begin{cases} 0 & \text{if } 0\le AP\le EX \\ 0 & \text{if } EX\le AP\le 2EX \end{cases}$$

$\Leftrightarrow P_m=0$

therefore $Z=1+0i=1$.

5-3- The Degree of Our Knowledge of the Random Experiment *DOK*

We have:

$$DOK=|Z|^2=P_r^2+(P_m/i)^2$$

$$=\begin{cases} \left(\dfrac{AP}{EX}\right)^2 & \text{if } 0\le AP\le EX \\ \left(2-\dfrac{AP}{EX}\right)^2 & \text{if } EX\le AP\le 2EX \end{cases} +\begin{cases} \left(1-\dfrac{AP}{EX}\right)^2 & \text{if } 0\le AP\le EX \\ \left(\dfrac{AP}{EX}-1\right)^2 & \text{if } EX\le AP\le 2EX \end{cases}$$

$$=\begin{cases} \left(\dfrac{AP}{EX}\right)^2+\left(1-\dfrac{AP}{EX}\right)^2 & \text{if } 0\le AP\le EX \\ \left(2-\dfrac{AP}{EX}\right)^2+\left(\dfrac{AP}{EX}-1\right)^2 & \text{if } EX\le AP\le 2EX \end{cases}$$

$$=\begin{cases} 2\left(\dfrac{AP}{EX}\right)^2-2\left(\dfrac{AP}{EX}\right)+1 & \text{if } 0\le AP\le EX \\ 2\left(\dfrac{AP}{EX}\right)^2-6\left(\dfrac{AP}{EX}\right)+5 & \text{if } EX\le AP\le 2EX \end{cases}$$

From *CPP* we have that $0.5\le DOK\le 1$ then if $DOK=0.5$

$$\Leftrightarrow\begin{cases} 2\left(\dfrac{AP}{EX}\right)^2-2\left(\dfrac{AP}{EX}\right)+1=0.5 & \text{if } 0\le AP\le EX \\ 2\left(\dfrac{AP}{EX}\right)^2-6\left(\dfrac{AP}{EX}\right)+5=0.5 & \text{if } EX\le AP\le 2EX \end{cases}$$

then solving the two second-degree equations for $\dfrac{AP}{EX}$ gives:

$$\begin{cases} \dfrac{AP}{EX}=1/2 & \text{if } 0\le AP\le EX \\ \dfrac{AP}{EX}=3/2 & \text{if } EX\le AP\le 2EX \end{cases} \Leftrightarrow \begin{cases} AP=EX/2 & \text{if } 0\le AP\le EX \\ AP=3EX/2 & \text{if } EX\le AP\le 2EX \end{cases}$$

and vice versa.

That means that *DOK* is minimum when the approximate result is equal to half of the exact result if $0 \le AP \le EX$ or when the approximate result is equal to three times the half of the exact result if $EX \le AP \le 2EX$, that means at the middle of the simulation.

In addition, if $DOK = 1$ then:

$$\Leftrightarrow \begin{cases} 2\left(\frac{AP}{EX}\right)^2 - 2\left(\frac{AP}{EX}\right) + 1 = 1 & \text{if } 0 \le AP \le EX \\ 2\left(\frac{AP}{EX}\right)^2 - 6\left(\frac{AP}{EX}\right) + 5 = 1 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(\frac{AP}{EX}\right)^2 - \left(\frac{AP}{EX}\right) = 0 & \text{if } 0 \le AP \le EX \\ 2\left(\frac{AP}{EX}\right)^2 - 6\left(\frac{AP}{EX}\right) + 4 = 0 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} AP = 0 \text{ OR } AP = EX & \text{if } 0 \le AP \le EX \\ AP = 2EX \text{ OR } AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$$

and vice versa.

That means that *DOK*, which is the degree of our knowledge of the random experiment, is maximum and is equal to 1 when the approximate result is equal to EX that means when it is equal to the exact result (at the end of the simulation) or 0 or $2EX$ (before the beginning of the simulation). We can deduce that we have total and perfect knowledge of the random experiment at the end of the simulation after the convergence of the numerical method to the exact result and hence when relative error is 0 = 0% and as well as before the beginning of the simulation since no randomness was introduced yet and thus when relative error is 1 = 100%.

5-4- The Chaotic Factor *Chf*

We have:

$$Chf = 2iP_rP_m = 2i \times \begin{cases} \frac{AP}{EX} & \text{if } 0 \le AP \le EX \\ 2 - \frac{AP}{EX} & \text{if } EX \le AP \le 2EX \end{cases} \times \begin{cases} i\left(1 - \frac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ i\left(\frac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

since $i^2 = -1$ then:

$$Chf = \begin{cases} -2\left(\frac{AP}{EX}\right)\left(1-\frac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ -2\left(2-\frac{AP}{EX}\right)\left(\frac{AP}{EX}-1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

From *CPP* we have that $-0.5 \le Chf \le 0$ then if $Chf = -0.5$

$$\Leftrightarrow \begin{cases} -2\left(\frac{AP}{EX}\right)\left(1-\frac{AP}{EX}\right) = -0.5 & \text{if } 0 \le AP \le EX \\ -2\left(2-\frac{AP}{EX}\right)\left(\frac{AP}{EX}-1\right) = -0.5 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} AP = EX/2 & \text{if } 0 \le AP \le EX \\ AP = 3EX/2 & \text{if } EX \le AP \le 2EX \end{cases}$$

and vice versa.

That means that *Chf* is minimum when the approximate result is equal to half of the exact result if $0 \le AP \le EX$ or when the approximate result is equal to three times the half of the exact result if $EX \le AP \le 2EX$, that means at the middle of the simulation.
In addition, if $Chf = 0$ then:

$$\Leftrightarrow \begin{cases} -2\left(\frac{AP}{EX}\right)\left(1-\frac{AP}{EX}\right) = 0 & \text{if } 0 \le AP \le EX \\ -2\left(2-\frac{AP}{EX}\right)\left(\frac{AP}{EX}-1\right) = 0 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} AP = 0 \text{ OR } AP = EX & \text{if } 0 \le AP \le EX \\ AP = 2EX \text{ OR } AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$$

And, conversely, if $\begin{cases} AP = 0 \text{ OR } AP = EX & \text{if } 0 \le AP \le EX \\ AP = 2EX \text{ OR } AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$ then $Chf = 0$.

That means that *Chf* is equal to 0 when the approximate result is equal to EX that means when it is equal to the exact result (at the end of the simulation) or 0 or $2EX$ (before the beginning of the simulation).

5-5- The Magnitude of the Chaotic Factor *MChf*

We have:

$$MChf = |Chf| = -2iP_rP_m$$

$$= -2i \times \begin{cases} \dfrac{AP}{EX} & \text{if } 0 \le AP \le EX \\ 2 - \dfrac{AP}{EX} & \text{if } EX \le AP \le 2EX \end{cases} \times \begin{cases} i\left(1 - \dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ i\left(\dfrac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

since $i^2 = -1$ then:

$$MChf = \begin{cases} 2\left(\dfrac{AP}{EX}\right)\left(1 - \dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ 2\left(2 - \dfrac{AP}{EX}\right)\left(\dfrac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

From *CPP* we have that $0 \le MChf \le 0.5$ then if $MChf = 0.5$

$$\Leftrightarrow \begin{cases} 2\left(\dfrac{AP}{EX}\right)\left(1 - \dfrac{AP}{EX}\right) = 0.5 & \text{if } 0 \le AP \le EX \\ 2\left(2 - \dfrac{AP}{EX}\right)\left(\dfrac{AP}{EX} - 1\right) = 0.5 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} AP = EX/2 & \text{if } 0 \le AP \le EX \\ AP = 3EX/2 & \text{if } EX \le AP \le 2EX \end{cases}$$

and vice versa.

That means that *MChf* is maximum when the approximate result is equal to half of the exact result if $0 \le AP \le EX$ or when the approximate result is equal to three times the half of the exact result if $EX \le AP \le 2EX$, that means at the middle of the simulation. This implies that the magnitude of the chaos (*MChf*) introduced by the variables used in the numerical method is maximum at the halfway of the simulation.

In addition, if $MChf = 0$ then:

$$\Leftrightarrow \begin{cases} 2\left(\dfrac{AP}{EX}\right)\left(1 - \dfrac{AP}{EX}\right) = 0 & \text{if } 0 \le AP \le EX \\ 2\left(2 - \dfrac{AP}{EX}\right)\left(\dfrac{AP}{EX} - 1\right) = 0 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \begin{cases} AP = 0 \text{ OR } AP = EX & \text{if } 0 \le AP \le EX \\ AP = 2EX \text{ OR } AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$$

And, conversely, if $\begin{cases} AP = 0 \text{ OR } AP = EX & \text{if } 0 \le AP \le EX \\ AP = 2EX \text{ OR } AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$ then $MChf = 0$.

That means that *MChf* is minimum and is equal to 0 when the approximate result is equal to EX that means when it is equal to the exact result (at the end of the simulation) or 0 or $2EX$ (before the beginning of the simulation). We can deduce that the magnitude of the chaos in the random experiment is null at the end of the simulation after the convergence of the numerical method to the exact result and when randomness has finished its task in the numerical method and experiment as well as before the beginning of the simulation since no randomness was introduced yet.

5-6- The Probability *Pc* in the Probability Set $\mathcal{C} = \mathcal{R} + \mathcal{M}$

We have from *CPP*:

$$Pc^2 = DOK - Chf = DOK + MChf$$

$$= \begin{cases} 2\left(\frac{AP}{EX}\right)^2 - 2\left(\frac{AP}{EX}\right) + 1 & \text{if } 0 \le AP \le EX \\ 2\left(\frac{AP}{EX}\right)^2 - 6\left(\frac{AP}{EX}\right) + 5 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$- \begin{cases} -2\left(\frac{AP}{EX}\right)\left(1 - \frac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ -2\left(2 - \frac{AP}{EX}\right)\left(\frac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} 1 & \text{if } 0 \le AP \le EX \\ 1 & \text{if } EX \le AP \le 2EX \end{cases} \Leftrightarrow Pc^2 = 1 \quad \text{for } 0 \le AP \le 2EX$$

$\Leftrightarrow Pc = 1$ = Probability and degree of convergence in $\mathcal{C}$, therefore:

$$Pc = \begin{cases} \frac{AP}{EX} = 1 & \text{if } 0 \le AP \le EX \\ 2 - \frac{AP}{EX} = 1 & \text{if } EX \le AP \le 2EX \end{cases} \Leftrightarrow \begin{cases} AP = EX & \text{if } 0 \le AP \le EX \\ AP = EX & \text{if } EX \le AP \le 2EX \end{cases}$$

$\Leftrightarrow AP = EX$ for $0 \le AP \le 2EX$ continuously in the probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$. This is due to the fact that in $\mathcal{C}$ we have subtracted in the equation above from our knowledge *DOK* the chaotic factor *Chf* and consequently we have removed chaos introduced and caused by all the variables and the numerical fluctuations that lead to approximate results in the numerical simulation in $\mathcal{R}$. Therefore, since in $\mathcal{C}$ we have always $AP = EX$ then the simulation which is a random method by nature in $\mathcal{R}$ becomes after applying the *CPP* a non-random method in $\mathcal{C}$ since the convergence probability of any experiment in $\mathcal{C}$ is permanently and constantly equal to 1 for any subintervals or iterations number *N*.

5-7- The Rates of Change of the Probabilities in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$

$$\text{Since } Z = P_r + P_m = \begin{cases} \dfrac{AP}{EX} + i\left(1 - \dfrac{AP}{EX}\right) & \text{if } 0 \le AP \le EX \\ \left(2 - \dfrac{AP}{EX}\right) + i\left(\dfrac{AP}{EX} - 1\right) & \text{if } EX \le AP \le 2EX \end{cases} = \mathrm{Re}(Z) + i\,\mathrm{Im}(Z)$$

Then:

$$\frac{dZ}{d(AP)} = \frac{dP_r}{d(AP)} + \frac{dP_m}{d(AP)} = \begin{cases} \dfrac{d}{d(AP)}\left[\dfrac{AP}{EX} + i\left(1 - \dfrac{AP}{EX}\right)\right] & \text{if } 0 \le AP \le EX \\ \dfrac{d}{d(AP)}\left[\left(2 - \dfrac{AP}{EX}\right) + i\left(\dfrac{AP}{EX} - 1\right)\right] & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} \dfrac{d}{d(AP)}\left[\dfrac{AP}{EX}\right] + \dfrac{d}{d(AP)}\left[i\left(1 - \dfrac{AP}{EX}\right)\right] & \text{if } 0 \le AP \le EX \\ \dfrac{d}{d(AP)}\left[2 - \dfrac{AP}{EX}\right] + \dfrac{d}{d(AP)}\left[i\left(\dfrac{AP}{EX} - 1\right)\right] & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} \dfrac{1}{EX} - \dfrac{i}{EX} = \dfrac{1}{EX}(1 - i) & \text{if } 0 \le AP \le EX \\ -\dfrac{1}{EX} + \dfrac{i}{EX} = \dfrac{1}{EX}(i - 1) & \text{if } EX \le AP \le 2EX \end{cases}$$

Therefore,

$$\mathrm{Re}\left[\frac{dZ}{d(AP)}\right] = \frac{dP_r}{d(AP)} = \begin{cases} +\dfrac{1}{EX} & \text{if } 0 \le AP \le EX \\ -\dfrac{1}{EX} & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} \text{constant} > 0 & \text{if } 0 \le AP \le EX \text{ and } EX > 0 \\ \text{constant} < 0 & \text{if } EX \le AP \le 2EX \text{ and } EX > 0 \end{cases}$$

that means that the rate of change or the slope of the probability of convergence in $\mathcal{R}$ is positive and constant if $0 \le AP \le EX$, and negative and constant if $EX \le AP \le 2EX$, and it depends only on $EX > 0$; hence, we have a constant increase in P_r (the convergence probability which is by definition an absolute value that means always nonnegative) as a function of the iterations or subintervals number *N* as AP increases from 0 to EX and as AP decreases from $2EX$ to EX till P_r reaches the value 1 that means till the random experiment converges to EX.

And $$= \begin{cases} \text{constant} < 0 & \text{if } EX \le AP \le 0 \text{ and } EX < 0 \\ \text{constant} > 0 & \text{if } 2EX \le AP \le EX \text{ and } EX < 0 \end{cases}$$

that means that the slope of the probability of convergence in $\mathcal{R}$ or its rate of change is constant and negative if $EX \le AP \le 0$, and constant and positive $2EX \le AP \le EX$, and it depends only on $EX < 0$; hence, we have a constant increase in P_r as a function of the iterations or subintervals number *N* as AP decreases from 0 to EX and as AP increases from $2EX$ to EX till P_r reaches the value 1 that means till the random experiment converges to EX .

$$\text{Im}\left[\frac{dZ}{d(AP)}\right] = \frac{1}{i}\frac{dP_m}{d(AP)} = \frac{d(P_m/i)}{d(AP)} = \begin{cases} -\dfrac{1}{EX} & \text{if } 0 \le AP \le EX \\ +\dfrac{1}{EX} & \text{if } EX \le AP \le 2EX \end{cases}$$

$$= \begin{cases} \text{constant} < 0 & \text{if } 0 \le AP \le EX \text{ and } EX > 0 \\ \text{constant} > 0 & \text{if } EX \le AP \le 2EX \text{ and } EX > 0 \end{cases}$$

that means that their rates of change or the slopes of the probabilities of divergence in $\mathcal{R}$ and $\mathcal{M}$ are negative and constant if $0 \le AP \le EX$, and positive and constant if $EX \le AP \le 2EX$, and they depend only on $EX > 0$; hence, we have a constant decrease in P_m/i and P_m (the divergence probabilities) as functions of the iterations or subintervals number *N* as AP increases from 0 to EX and as AP decreases from $2EX$ to EX till P_m/i and P_m reach the value 0 that means till the random experiment converges to EX .

And $$= \begin{cases} \text{constant} > 0 & \text{if } EX \le AP \le 0 \text{ and } EX < 0 \\ \text{constant} < 0 & \text{if } 2EX \le AP \le EX \text{ and } EX < 0 \end{cases}$$

that means that the slopes of the probabilities of divergence in $\mathcal{R}$ and $\mathcal{M}$ or their rates of change are constant and positive if $EX \le AP \le 0$, and constant and negative if $2EX \le AP \le EX$, and they depend only on $EX < 0$; hence, we have a constant decrease in P_m/i and P_m as functions of the iterations or subintervals number *N* as AP decreases from 0 to EX and as AP increases from $2EX$ to EX till P_m/i and P_m reach the value 0 that means till the random experiment converges to EX .

Additionally,

$$\left|\frac{dZ}{d(AP)}\right|^2 = \left[\frac{dP_r}{d(AP)}\right]^2 + \left[\frac{1}{i}\times\frac{dP_m}{d(AP)}\right]^2 = \left[\frac{dP_r}{d(AP)}\right]^2 + \left[\frac{d(P_m/i)}{d(AP)}\right]^2$$

$$= \begin{cases} \left(\dfrac{1}{EX}\right)^2 + \left(-\dfrac{1}{EX}\right)^2 & \text{if } 0 \le AP \le EX \\ \left(-\dfrac{1}{EX}\right)^2 + \left(\dfrac{1}{EX}\right)^2 & \text{if } EX \le AP \le 2EX \end{cases}$$

$$\Leftrightarrow \left|\frac{dZ}{d(AP)}\right|^2 = \frac{1}{(EX)^2} + \frac{1}{(EX)^2} = \frac{2}{(EX)^2} \quad \text{for } 0 \le AP \le 2EX$$

$$\Leftrightarrow \left|\frac{dZ}{d(AP)}\right| = \frac{\sqrt{2}}{|EX|} = \text{constant} > 0, \ \forall EX \ ;$$

that means that its rate of change or the module of the slope of the complex probability vector *Z* in $\mathcal{C}$ is positive and constant and it depends only on $|EX|$; hence, we have a constant increase in $\text{Re}(Z)$ and a constant decrease in $\text{Im}(Z)$ as functions of the iterations or subintervals number *N* and as *Z* goes from (0, *i*) at *N* = 0 till (1,0) at the simulation end; hence, till $\text{Re}(Z) = P_r$ reaches the value 1 that means till the random experiment converges to EX .

Furthermore, since $Pc^2 = DOK - Chf = DOK + MChf = 1$ from *CPP*

then $Pc = 1 =$ Probability and degree of convergence in $\mathcal{C}$

and consequently: $\frac{d(Pc)}{d(AP)} = \frac{d(1)}{d(AP)} = 0$;

that means that *Pc* is constantly equal to 1 for every value of AP , of EX , and of the iterations or subintervals number *N*, that means for any random experiment and for any simulation of the numerical methods. So, we conclude that in $\mathcal{C}$ we have complete and perfect knowledge of the random experiment which has become now a non-random one since the extension in the complex probability plane $\mathcal{C}$ defined by the *CPP* axioms has changed all random variables to non-random variables and since we have subtracted and eliminated in the equation of *Pc* above chaos expressed by *Chf* from *DOK*. Hence, randomness and chaos vanish completely and totally in the probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$.

6- The Evaluation of the New Paradigm Parameters

We can deduce from what has been elaborated previously the following:

The real convergence probability: $P_r(N) = 1 - \left|\frac{EX - AP(N)}{EX}\right|$

We have $0 \le N \le N_C$ where *N* = 0 corresponds to the instant before the beginning of the random experiment when $AP(N = 0) = 0$ or $= 2EX$, and where $N = N_C$ (iterations number needed for the method convergence) corresponds to the instant at the end of the random experiments and Monte Carlo methods when $AP(N = N_C) \to EX$.

The imaginary complementary divergence probability: $P_m(N) = i\left|\frac{EX - AP(N)}{EX}\right|$

The real complementary divergence probability: $P_m(N)/i = \left|\frac{EX - AP(N)}{EX}\right|$

The complex probability and random vector:

$$Z(N) = P_r(N) + P_m(N) = \left[1 - \left|\frac{EX - AP(N)}{EX}\right|\right] + i\left|\frac{EX - AP(N)}{EX}\right|$$

The Degree of Our Knowledge:

$$DOK(N) = |Z(N)|^2 = P_r^2(N) + [P_m(N)/i]^2 = \left[1 - \left|\frac{EX - AP(N)}{EX}\right|\right]^2 + \left[\left|\frac{EX - AP(N)}{EX}\right|\right]^2$$

$$= 1 + 2iP_r(N)P_m(N) = 1 - 2P_r(N)[1 - P_r(N)] = 1 - 2P_r(N) + 2P_r^2(N)$$

$$= 1 - 2\left|\frac{EX - AP(N)}{EX}\right| + 2\left[\frac{EX - AP(N)}{EX}\right]^2$$

$DOK(N)$ is equal to 1 when $P_r(N) = P_r(0) = 0$ and when $P_r(N) = P_r(N_C) = 1$.

The Chaotic Factor:

$$Chf(N) = 2iP_r(N)P_m(N) = -2P_r(N)[1 - P_r(N)] = -2P_r(N) + 2P_r^2(N)$$

$$= -2\left|\frac{EX - AP(N)}{EX}\right| + 2\left[\frac{EX - AP(N)}{EX}\right]^2$$

$Chf(N)$ is null when $P_r(N) = P_r(0) = 0$ and when $P_r(N) = P_r(N_C) = 1$.

The Magnitude of the Chaotic Factor *MChf*:

$$MChf(N) = |Chf(N)| = -2iP_r(N)P_m(N) = 2P_r(N)[1 - P_r(N)] = 2P_r(N) - 2P_r^2(N)$$

$$= 2\left|\frac{EX - AP(N)}{EX}\right| - 2\left[\frac{EX - AP(N)}{EX}\right]^2$$

$MChf(N)$ is null when $P_r(N) = P_r(0) = 0$ and when $P_r(N) = P_r(N_C) = 1$.

At any iteration number *N*: $0 \le \forall N \le N_C$, the probability expressed in the complex probability set $\mathcal{C}$ is the following:

$$Pc^2(N) = [P_r(N) + P_m(N)/i]^2 = |Z(N)|^2 - 2iP_r(N)P_m(N)$$

$$= DOK(N) - Chf(N)$$

$$= DOK(N) + MChf(N)$$

$$= 1$$

then,

$$Pc^2(N) = [P_r(N) + P_m(N)/i]^2 = \{P_r(N) + [1 - P_r(N)]\}^2 = 1^2 = 1 \Leftrightarrow Pc(N) = 1 \text{ always}$$

Hence, the prediction of the convergence probabilities of the stochastic the neutron shielding experiment in the set $\mathcal{C}$ is permanently certain.

Let us consider thereafter a multidimensional integral and a stochastic experiment to simulate the neutron shielding method and to draw, to visualize, as well as to quantify all the *CPP* and prognostic parameters.

7- The C++ Algorithms of the Neutron Shielding Method

7-1- The First Algorithm with the C++ Built-in Uniform Random Number Generator

```
// The Neutron Shielding Algorithm with the C++ Built-in Uniform
// Random Number Generator

#include <iostream>
#include <cstdlib>
#include <ctime>
#include <cmath>
#include <iomanip>

using namespace std;

const long double PI = 3.1415926535897931;

int main()
{
      long int i, j, N, Nc;
      long double d, alpha, counter, AP;

      cout << "        THE NEUTRON SHIELDING PROGRAM" << endl;
      cout <<
       "        ------------------------------------------------------"
            << "\n" << endl;

      for (i = 1; i <= 10; i++)
      {
            srand(time(0));

            Nc = 100000000; counter = 0;
            for (N = 1; N <= Nc; N++)
            {
                  d = 1; j = 1;
                  while ((j <= 10) && (d > 0) && (d < 5))
                  {
                        alpha = 2 * PI * (long double) rand() / 32767;
                        d = d + (long double) cos(alpha);
                        alpha = 0;
                        j++;
                  }
```

```
                if (d >= 5)
                      ++counter;
          }

          AP = 100 * (long double) counter / Nc;
          cout << fixed << setprecision(0) << "AFTER A SIMULATION OF "
               << Nc << " PARTICLES," << "\n" << "ONLY "
               << fixed << setprecision(5) << AP
               << "% OF THE NEUTRONS EMERGED FROM THE LEAD WALL"
               << "\n" << endl;
     }

     return 0;
}
```

7-2- The Second Algorithm with a Second Uniform Random Number Generator

```
// The Neutron Shielding Algorithm with Another Second Uniform Random
// Number Generator

#include <iostream>
#include <cstdlib>
#include <ctime>
#include <cmath>
#include <iomanip>

using namespace std;

long double generate(long double, long double);
long double total();

const long double PI = 3.1415926535897931;
const long double BOUND = 2147483647;

int main()
{
     long double Nc, summation;
     int c;

     cout << "        THE NEUTRON SHIELDING PROGRAM" << endl;
     cout <<
      "        ----------------------------------------------------"
          << "\n" << endl;

     Nc = 100000000;
```

```
	for (c = 1; c <= 10; c++)
	{
		summation = (long double) total();
		cout << fixed << setprecision(0) << "AFTER A SIMULATION OF "
		     << Nc << " PARTICLES," << "\n" << "ONLY "
		     << fixed << setprecision(5) << summation
		     << "% OF THE NEUTRONS EMERGED FROM THE LEAD WALL"
		     << "\n" << endl;
	}

	return 0;
}

long double gen(long double xn1, long double xn2)
{
	long double xn;

	xn = (long double) fmod(((1999 * xn1) + (4444 * xn2)), BOUND);

	return xn;
}

long double total(void)
{
	long int N, Nc, counter = 0;
	long double sxn1 = 0, sxn2 = 0, d = 0, j = 0, random = 0,
		    alpha = 0, AP = 0;

	srand(time(0));
	sxn1 = rand();
	sxn2 = rand();

	Nc = 100000000;
	for (N = 1; N <= Nc; N++)
	{
		d = 1; j = 1;
		while ((j <= 10) && (d > 0) && (d < 5))
		{
			random = gen(sxn1, sxn2);
			alpha = 2 * PI * (long double) random / BOUND;
			d = d + (long double) cos(alpha);
			alpha = 0;
			sxn2 = sxn1;
			sxn1 = random;
			j++;
		}
```

```
            if (d >= 5)
                  ++counter;
      }

      AP = 100 * (long double) counter / Nc;

      return AP;
}
```

7-3- The Third Algorithm with a Third Uniform Random Number Generator

```
// The Neutron Shielding Algorithm with Another Third Uniform Random
// Number Generator

#include <iostream>
#include <cstdlib>
#include <ctime>
#include <cmath>
#include <iomanip>

using namespace std;

long double generate(long double, long double);
long double total();

const long double PI = 3.1415926535897931;
const long double BOUND = 2147483647;

int main()
{
      long double Nc, summation;
      int c;

      cout << "       THE NEUTRON SHIELDING PROGRAM" << endl;
      cout <<
        "       ----------------------------------------------------"
           << "\n" << endl;

      Nc = 100000000;
      for (c = 1; c <= 10; c++)
      {
            summation = (long double) total();

            cout << fixed << setprecision(0) << "AFTER A SIMULATION OF "
                 << Nc << " PARTICLES," << "\n" << "ONLY "
```

```
                << fixed << setprecision(5) << summation
                << "% OF THE NEUTRONS EMERGED FROM THE LEAD WALL"
                << "\n" << endl;
        }
        return 0;
}

long double gen(long double xn1)
{
        long double xn;
        xn = (long double) fmod(((69069 * xn1) + 1), BOUND);
        return xn;
}

long double total(void)
{
        long int N, Nc, counter = 0;
        long double sxn1 = 0, d = 0, j = 0, random = 0, alpha = 0,
                    AP = 0;

        srand(time(0));
        sxn1 = rand();

        Nc = 100000000;
        for (N = 1; N <= Nc; N++)
        {
                d = 1; j = 1;
                while ((j <= 10) && (d > 0) && (d < 5))
                {
                        random = gen(sxn1);
                        alpha = 2 * PI * (long double) random / BOUND;
                        d = d + (long double) cos(alpha);
                        alpha = 0;
                        sxn1 = random;
                        j++;
                }

                if (d >= 5)
                        ++counter;
        }

        AP = 100 * (long double) counter / Nc;

        return AP;
}
```

8- Flowchart of the Complex Probability and the Neutron Shielding Prognostic Model

The following flowchart summarizes all the procedures of the proposed complex probability prognostic model:

9- Simulation of the New Paradigm

Note that all the numerical values found in the simulations of the new paradigm for any iteration cycles *N* were computed using the 64-Bit MATLAB version 2024 software. In addition, the reader should take care of the rounding and truncation errors since all numerical values in the solution of the problem are represented by at most five significant digits and since we are using the neutron shielding method of simulation which gives approximate results subject to random effects and fluctuations. We have considered for this purpose a high-capacity computer system: a workstation computer with parallel microprocessors, a 64-Bit operating system, and a 64-GB RAM. Additionally, we have replaced in all the simulations $AP(N)$ by $AP(N)/4$ and EX by $EX/4$ to better see and read the simulations and to fit all the data and figures in a nicer and improved view.

9-1- The Uniform Random Numbers Generator Case

We will use in the first case in the solution of the neutron shielding problem the uniform random numbers generator: $(random, \alpha, d) \mapsto \mathbb{U}\ (0,10)$

$\Leftrightarrow AP(N) = \frac{1}{4} \times 100 \times \frac{counter}{N}\%$ with $1 \le N \le N_C$ after applying the neutron shielding method.

Moreover, the three figures (Figures 5-7) show the increasing convergence of the neutron shielding method and simulation to the exact result $EX \cong 5.1875/4 \cong 1.296875 \cong 1.2969\%$ for *N* = 1000, 30000, and $N_C = 400000$ iterations. Therefore, we have:

$\lim_{N \to +\infty} P_r(N) = \lim_{N \to +\infty} \left\{ 1 - \left| \frac{EX - AP(N)}{EX} \right| \right\} = 1 - \left| \frac{EX - EX}{EX} \right| = 1 - 0 = 1$ which is equal to the convergence probability of the neutron shielding method as $N \to +\infty$.

Additionally, Figure 8 illustrates clearly and visibly the relation of the neutron shielding method to the complex probability paradigm with all its parameters ($Chf, MChf, DOK, EX, AP, P_r, P_m / i, Pc$).

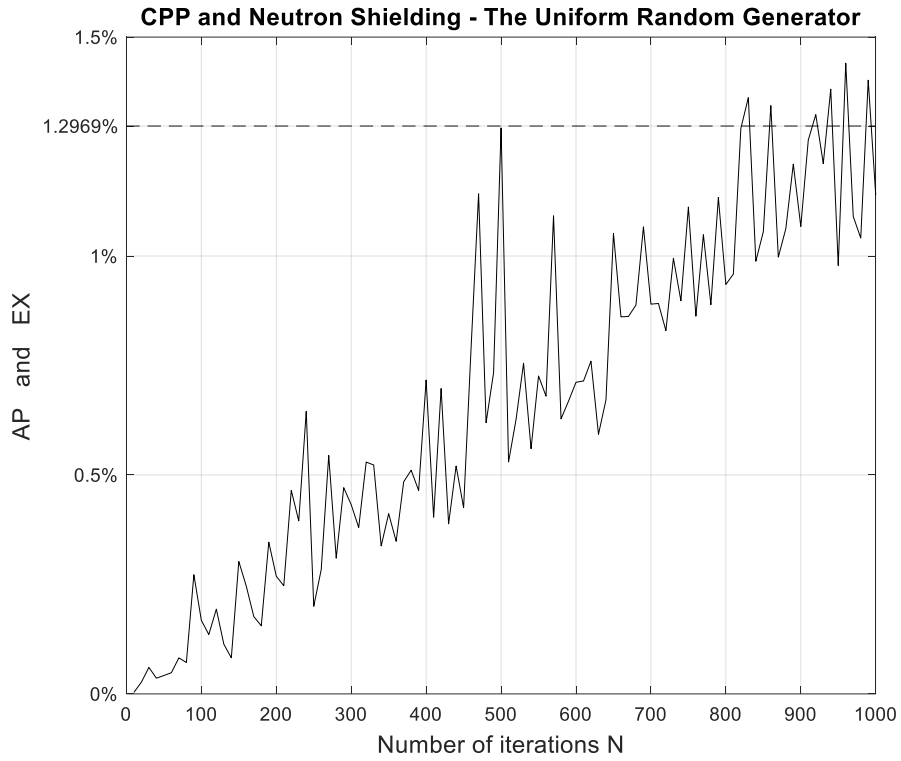


Figure 5- The increasing convergence of the neutron shielding algorithm up to N = 1000 iterations with the Uniform random number generator.

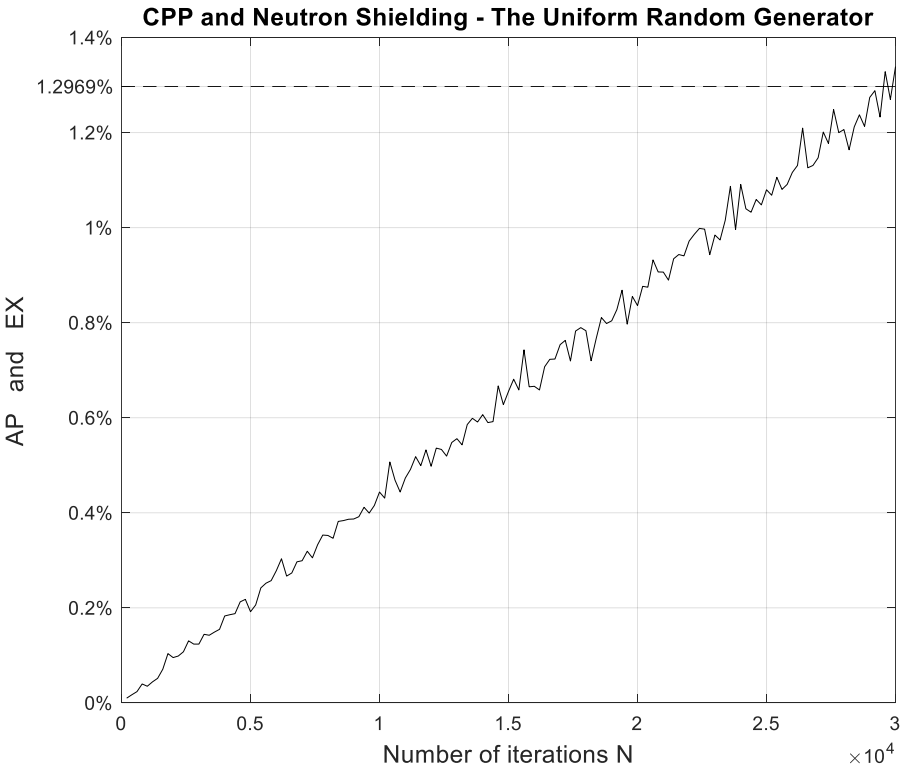


Figure 6- The increasing convergence of the neutron shielding algorithm up to N = 30,000 iterations with the Uniform random number generator.

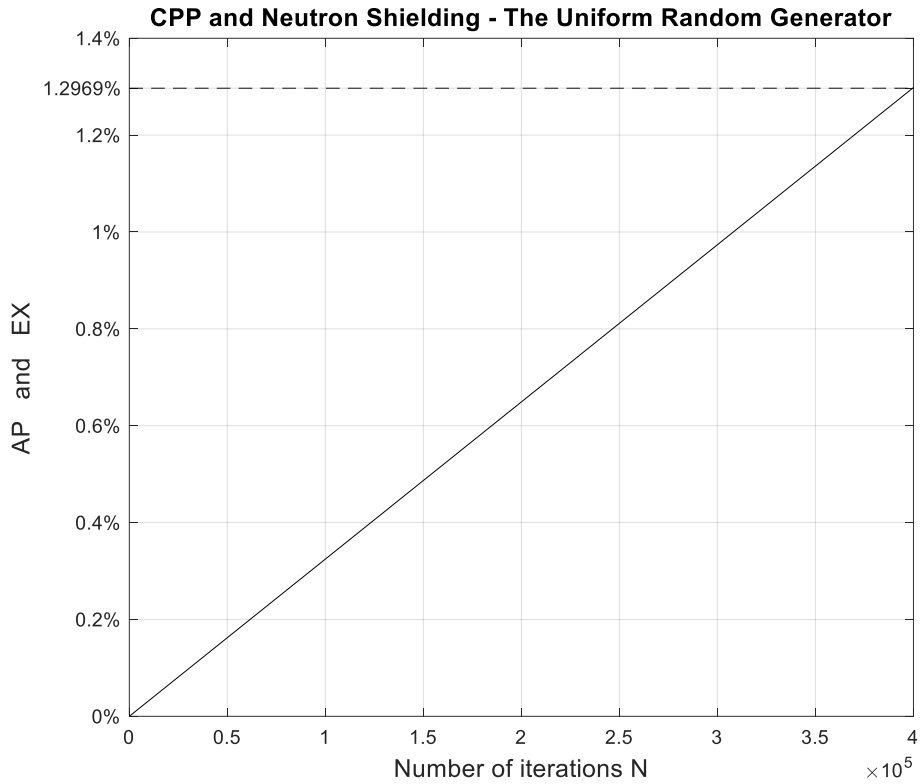


Figure 7- The increasing convergence of the neutron shielding algorithm up to N = 400,000 iterations with the Uniform random number generator.

VII-2-1 The Complex Probability Cubes

In the first cube (Figure 9), the simulation of *DOK* and *Chf* as functions of each other and of the iterations N for the neutron shielding problem can be seen. The thick line in cyan is the projection of $Pc^2(N) = DOK(N) - Chf(N) = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point J ($DOK = 1$, $Chf = 0$) when $N = 0$ iterations, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when N = 200,000 iterations, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $N = N_C$ = 400,000 iterations. The other curves are the graphs of $DOK(N)$ (red) and $Chf(N)$ (green, blue, pink) in different planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, N = 200,000 iterations). The point L corresponds to ($DOK = 1$, $Chf = 0$, $N = N_C$ = 400,000 iterations). The three points J, K, L are the same as in Figure 8.

In the second cube (Figure 10), we can notice the simulation of the convergence probability $P_r(N)$ and its complementary real divergence probability $P_m(N)/i$ in terms of the iterations N for the neutron shielding problem. The thick line in cyan is the projection of $Pc^2(N) = P_r(N) + P_m(N)/i = 1 = Pc(N)$ on the plane $N = 0$ iterations. This line starts at the point ($P_r = 0$, $P_m/i = 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $P_r(N)$ in the plane $P_r(N) = P_m(N)/i$. This curve starts at the point J ($P_r = 0$, $P_m/i = 1$, $N = 0$ iterations), reaches the point K ($P_r = 0.5$, $P_m/i = 0.5$, N = 200,000 iterations), and gets at the end to L ($P_r = 1$, $P_m/i = 0$, $N = N_C$ = 400,000 iterations). The blue curve represents $P_m(N)/i$ in the plane $P_r(N) + P_m(N)/i = 1$. Notice the importance of the point K which is the intersection of the red and blue curves at N = 200,000 iterations and when $P_r(N) = P_m(N)/i = 0.5$. The three points J, K, L are the same as in Figure 8.

In the third cube (Figure 11), we can notice the simulation of the complex random vector $Z(N)$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ as a function of the real convergence probability $P_r(N) = \mathrm{Re}(Z)$ in $\mathcal{R}$ and of its complementary imaginary divergence probability $P_m(N) = i \times \mathrm{Im}(Z)$ in $\mathcal{M}$, and this in terms of the iterations N for the neutron shielding problem. The red curve represents $P_r(N)$ in the plane $P_m(N) = 0$ and the blue curve represents $P_m(N)$ in the plane $P_r(N) = 0$. The green curve represents the complex probability vector $Z(N) = P_r(N) + P_m(N) = \mathrm{Re}(Z) + i \times \mathrm{Im}(Z)$ in the plane $P_r(N) = iP_m(N) + 1$. The curve of $Z(N)$ starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point L ($P_r = 1$, $P_m = 0$, $N = N_C = 400{,}000$ iterations). The thick line in cyan is $P_r(0) = iP_m(0) + 1$ and it is the projection of the $Z(N)$ curve on the complex probability plane whose equation is $N = 0$ iterations. This projected line starts at the point J ($P_r = 0$, $P_m = i$, $N = 0$ iterations) and ends at the point ($P_r = 1$, $P_m = 0$, $N = 0$ iterations). Notice the importance of the point K corresponding to $N = 200{,}000$ iterations and when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in Figure 8.

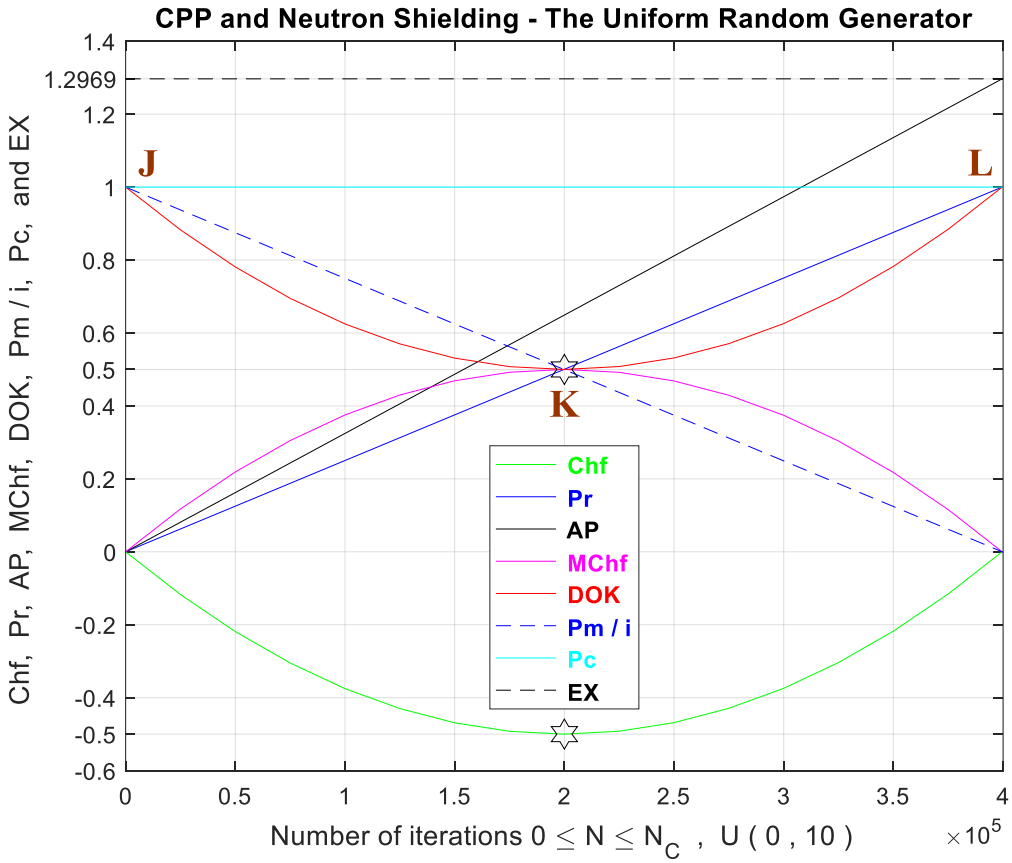


Figure 8- The *CPP* parameters and the neutron shielding algorithm with the Uniform random number generator.

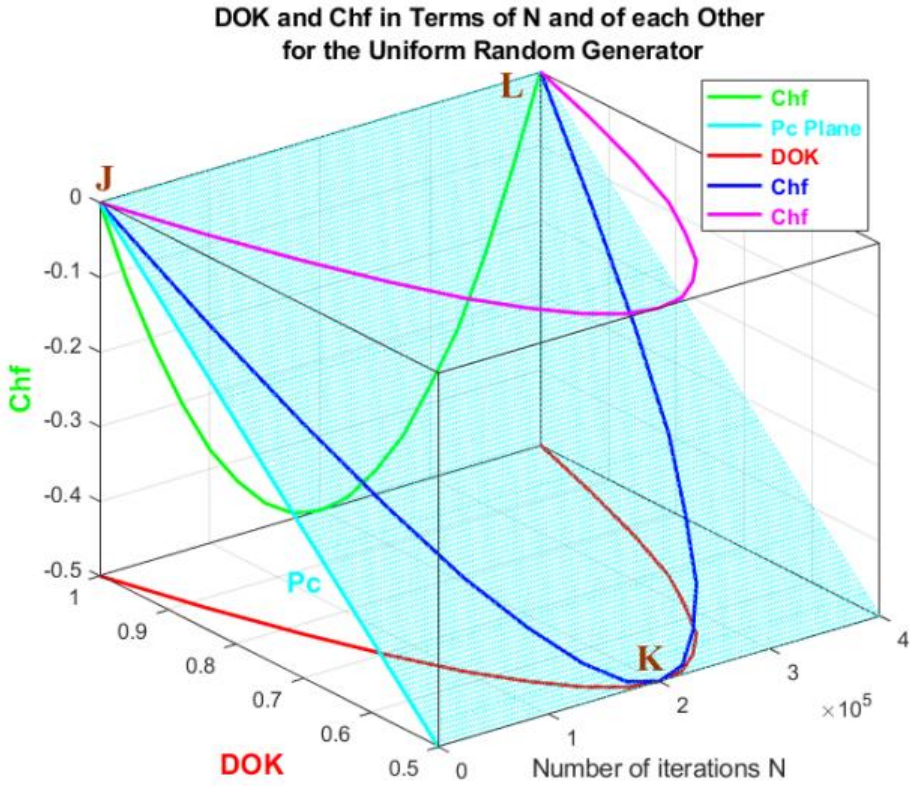


Figure 9- *DOK* and *Chf* in terms of *N* and of each other for the neutron shielding algorithm with the Uniform random number generator.

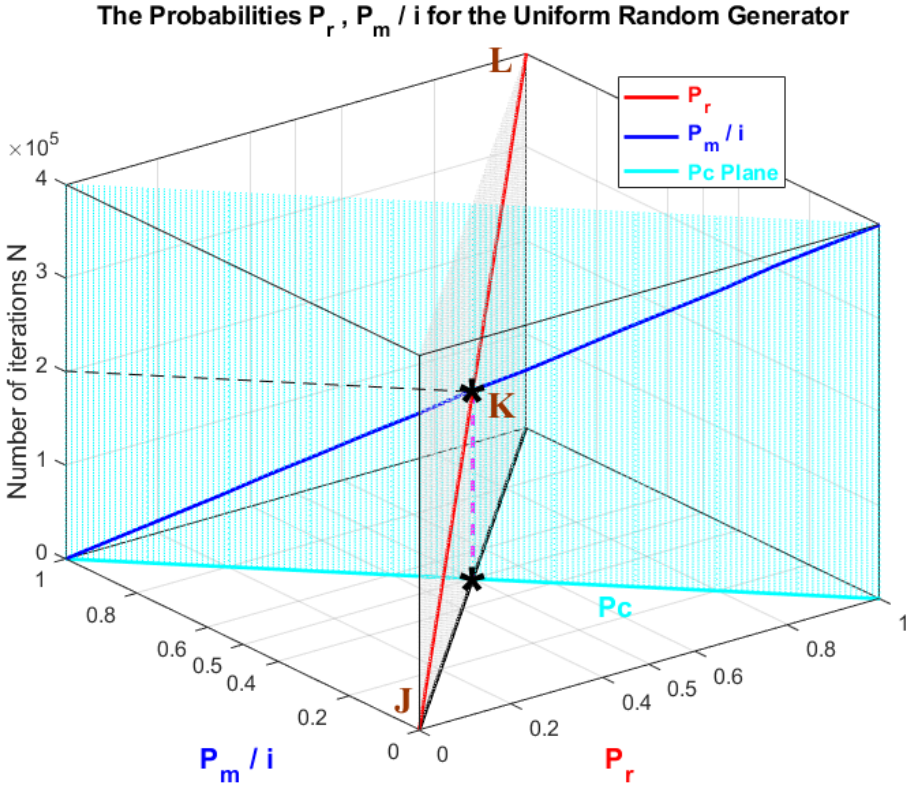


Figure 10- P_r and P_m/i in terms of *N* and of each other for the neutron shielding algorithm with the Uniform random number generator.

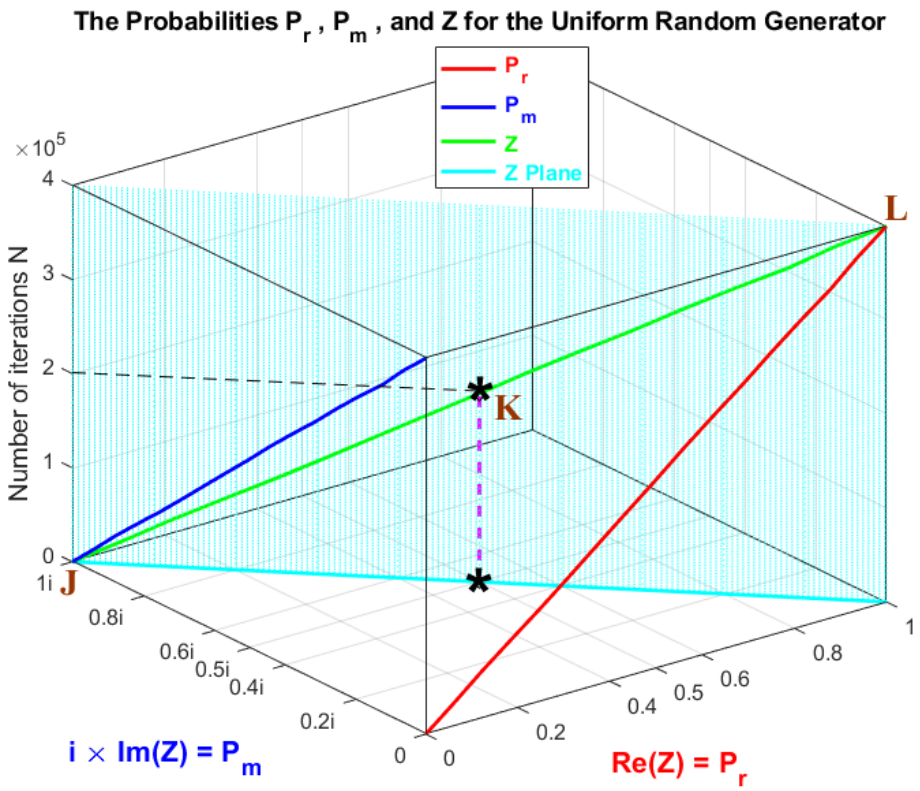


Figure 11- The Complex Probability Vector *Z* in terms of *N* for the neutron shielding algorithm with the Uniform random number generator.

9-2- The Gaussian and Normal Random Numbers Generator Case

We will use in the second case in the solution of the neutron shielding problem the Gaussian and normal random numbers generator:

$$(random, \alpha, d) \mapsto \mathbb{N}\ (\mu = 0, \sigma = 1)$$

$$\Leftrightarrow AP(N) = \frac{1}{4} \times 100 \times \frac{counter}{N}\% \text{ with } 1 \le N \le N_C \text{ after applying the neutron shielding method.}$$

Moreover, the three figures (Figures 12-14) show the increasing convergence of the neutron shielding method and simulation to the exact result $EX \cong 5.1875/4 \cong 1.296875 \cong 1.2969\%$ for N = 1000, 30000, and $N_C = 400000$ iterations. Therefore, we have:

$$\lim_{N \to +\infty} P_r(N) = \lim_{N \to +\infty} \left\{ 1 - \left| \frac{EX - AP(N)}{EX} \right| \right\} = 1 - \left| \frac{EX - EX}{EX} \right| = 1 - 0 = 1$$ which is equal to the convergence probability of the neutron shielding method as $N \to +\infty$.

Additionally, Figure 15 illustrates clearly and visibly the relation of the neutron shielding method to the complex probability paradigm with all its parameters $(Chf, MChf, DOK, EX, AP, P_r, P_m / i, Pc)$ after applying it to this method.

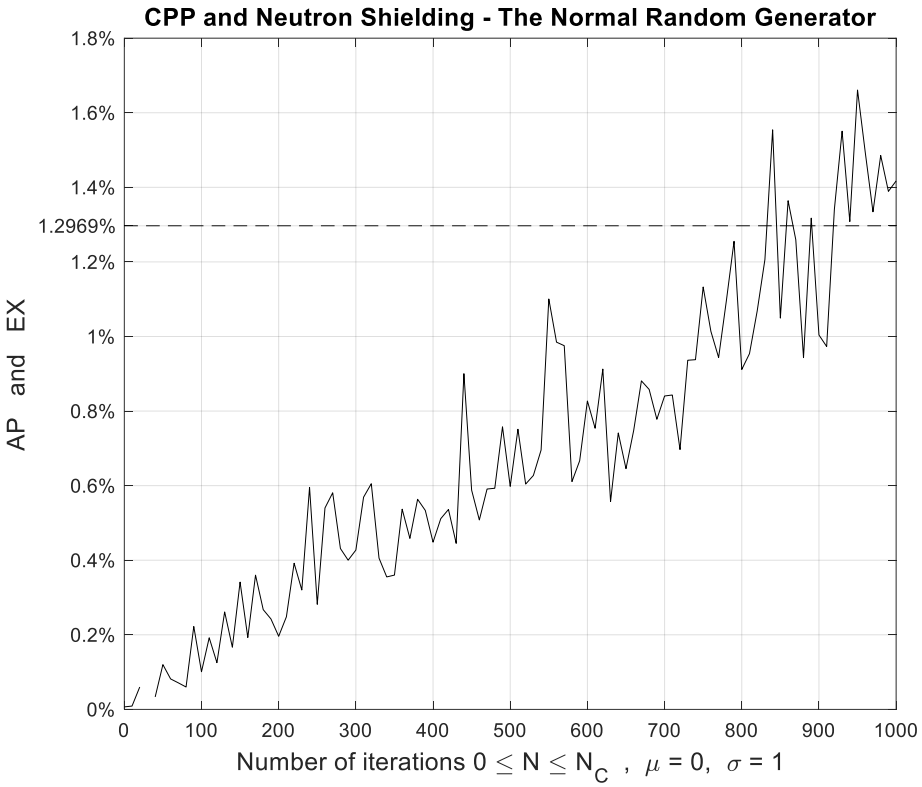


Figure 12- The increasing convergence of the neutron shielding algorithm up to N = 1000 iterations with the Gaussian and normal random number generator.

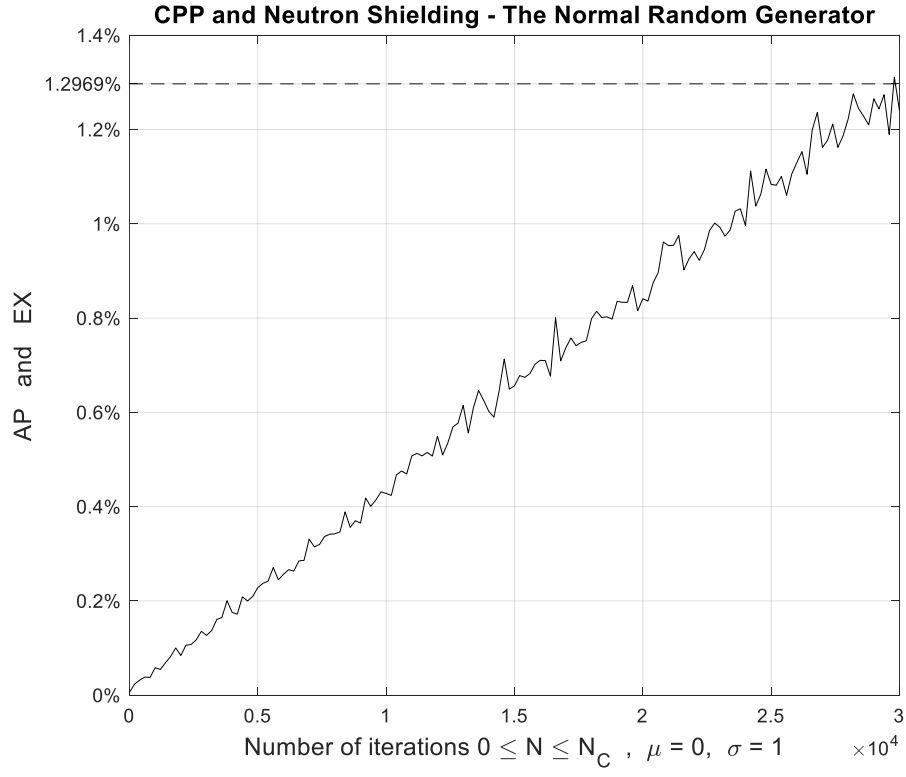


Figure 13- The increasing convergence of the neutron shielding algorithm up to N = 30,000 iterations with the Gaussian and normal random number generator.

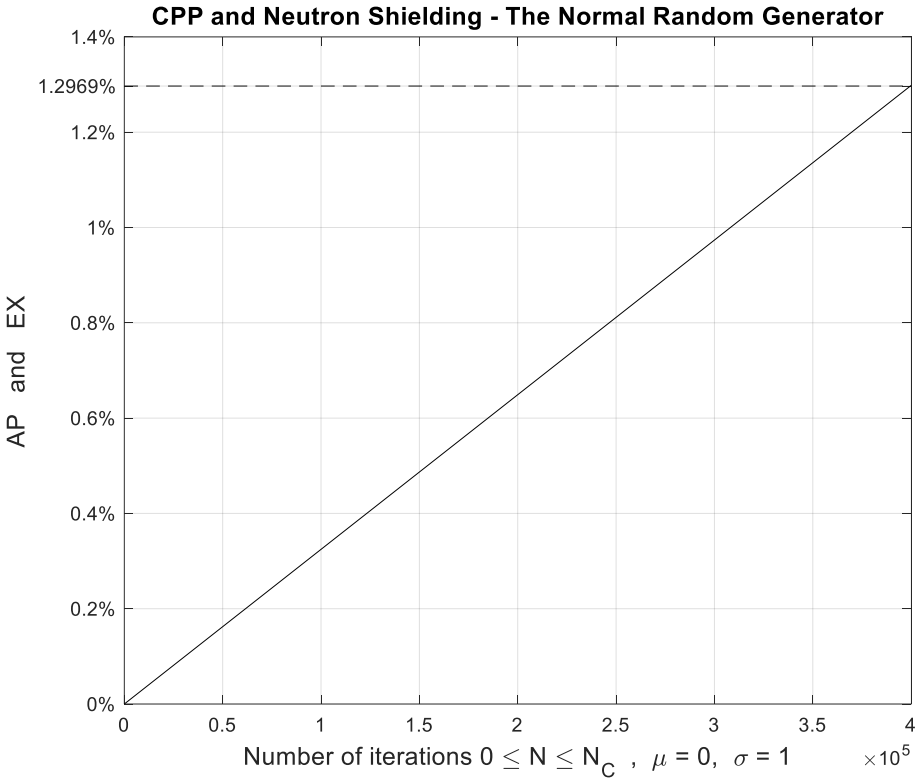


Figure 14- The increasing convergence of the neutron shielding algorithm up to N = 400,000 iterations with the Gaussian and normal random number generator.

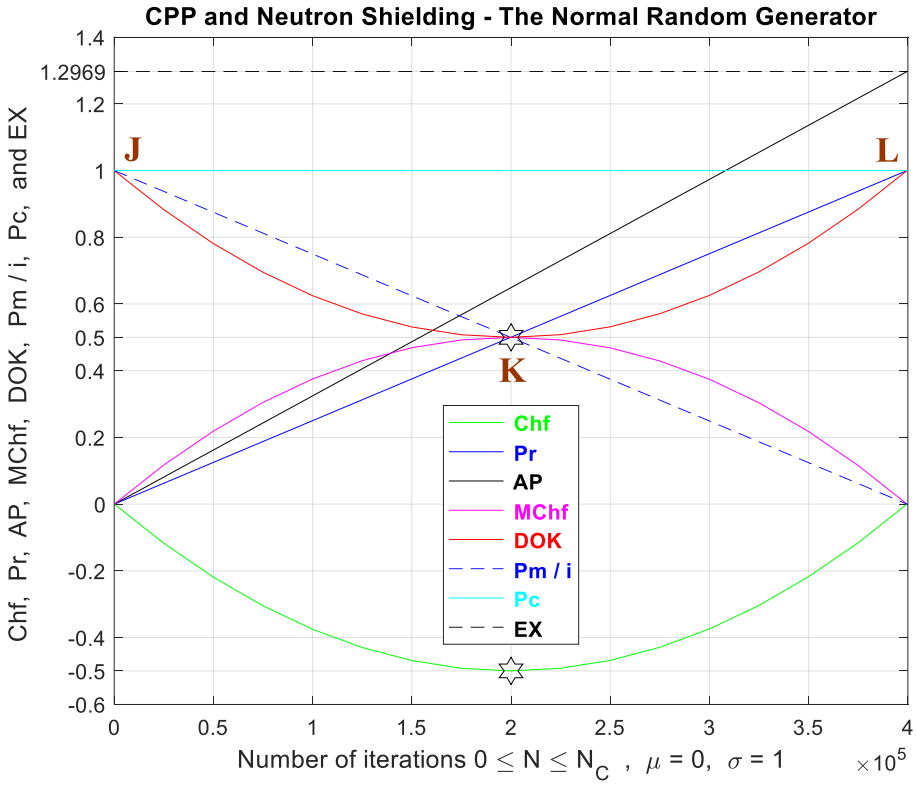


Figure 15- The *CPP* parameters and the neutron shielding algorithm with the Gaussian and normal random number generator.

9-3- The Poisson Random Numbers Generator Case

We will use in the third case in the solution of the neutron shielding problem the Poisson random numbers generator:

$$(random, \alpha, d) \mapsto \mathbb{P} \ (\lambda = 5.68)$$

$\Leftrightarrow AP(N) = \frac{1}{4} \times 100 \times \frac{counter}{N}\%$ with $1 \le N \le N_C$ after applying the neutron shielding method.

Moreover, the three figures (Figures 16-18) show the increasing convergence of the neutron shielding method and simulation to the exact result $EX \cong 5.1875/4 \cong 1.296875 \cong 1.2969\%$ for N = 1000, 30000, and $N_C = 400000$ iterations. Therefore, we have:

$\lim_{N \to +\infty} P_r(N) = \lim_{N \to +\infty} \left\{ 1 - \left| \frac{EX - AP(N)}{EX} \right| \right\} = 1 - \left| \frac{EX - EX}{EX} \right| = 1 - 0 = 1$ which is equal to the convergence probability of the neutron shielding method as $N \to +\infty$.

Additionally, Figure 19 illustrates clearly and visibly the relation of the neutron shielding method to the complex probability paradigm with all its parameters ($Chf, MChf, DOK, EX, AP, P_r, P_m / i, Pc$) after applying it to this method.

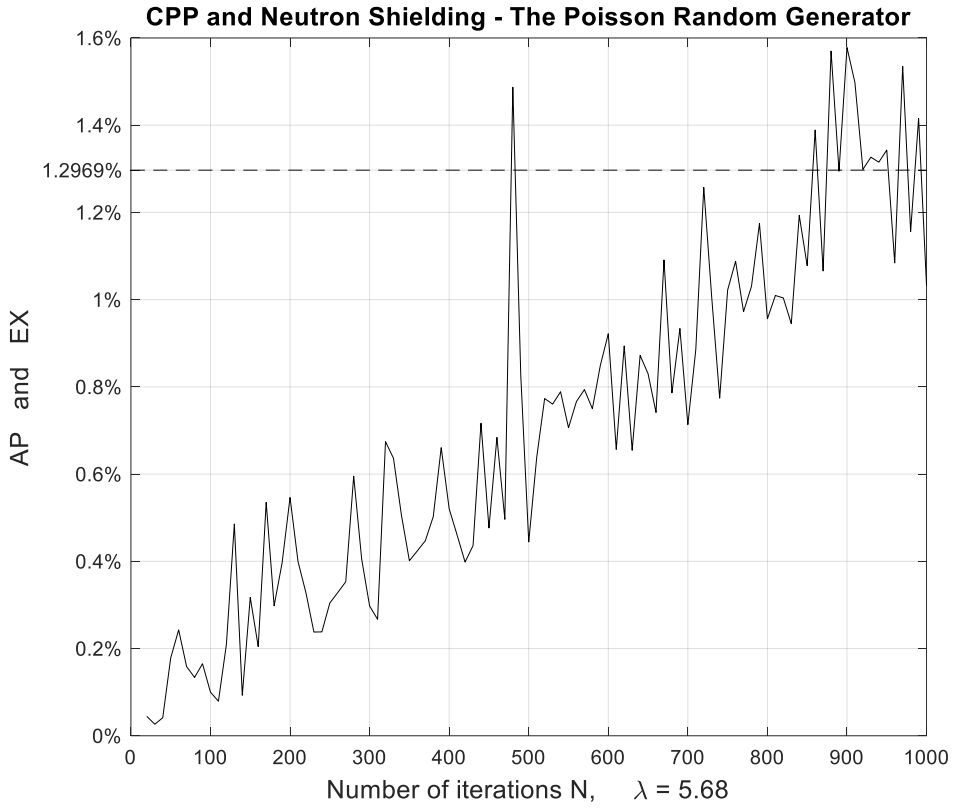


Figure 16- The increasing convergence of the neutron shielding algorithm up to N = 1000 iterations with the Poisson random number generator.

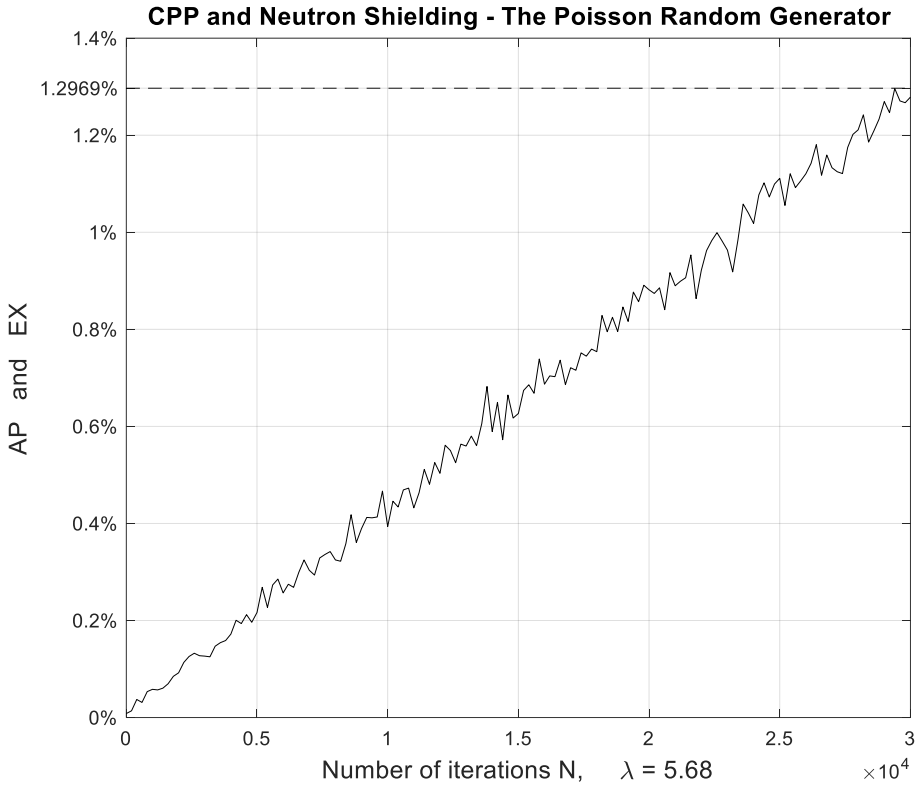


Figure 17- The increasing convergence of the neutron shielding algorithm up to N = 30,000 iterations with the Poisson random number generator.

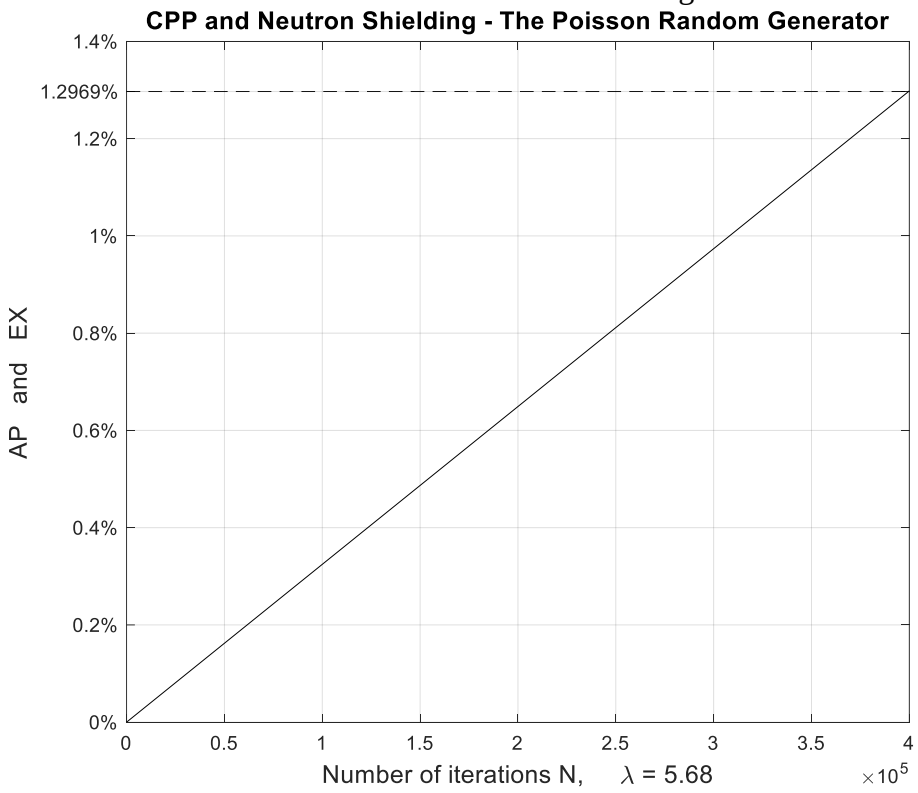


Figure 18- The increasing convergence of the neutron shielding algorithm up to N = 400,000 iterations with the Poisson random number generator.

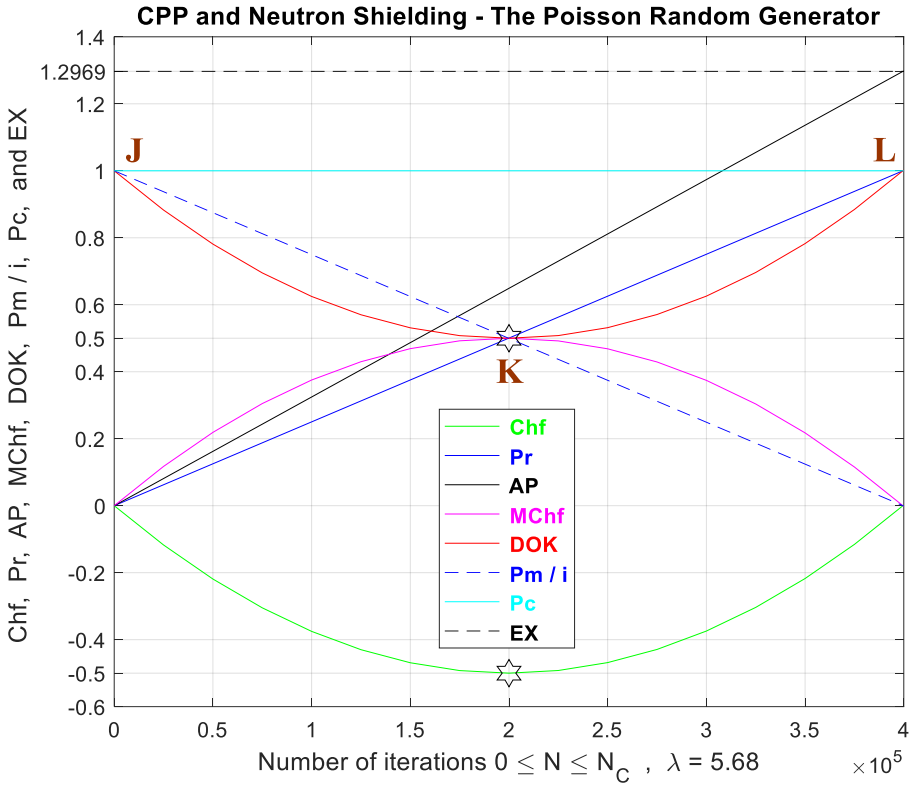


Figure 19- The *CPP* parameters and the neutron shielding algorithm with the Poisson random number generator.

9-4- The Weibull Random Numbers Generator Case

We will use in the fourth case in the solution of the neutron shielding problem the Weibull random numbers generator:

$$(random, \alpha, d) \mapsto \mathbb{W} \ (a = 1, b = 2)$$

$\Leftrightarrow AP(N) = \frac{1}{4} \times 100 \times \frac{counter}{N}\%$ with $1 \le N \le N_C$ after applying the neutron shielding method.

Moreover, the three figures (Figures 20-22) show the increasing convergence of the neutron shielding method and simulation to the exact result $EX \cong 5.1875/4 \cong 1.296875 \cong 1.2969\%$ for N = 1000, 30000, and $N_C = 400000$ iterations. Therefore, we have:

$\lim_{N \to +\infty} P_r(N) = \lim_{N \to +\infty} \left\{ 1 - \left| \frac{EX - AP(N)}{EX} \right| \right\} = 1 - \left| \frac{EX - EX}{EX} \right| = 1 - 0 = 1$ which is equal to the convergence probability of the neutron shielding method as $N \to +\infty$.

Additionally, Figure 23 illustrates clearly and visibly the relation of the neutron shielding method to the complex probability paradigm with all its parameters $(Chf, MChf, DOK, EX, AP, P_r, P_m / i, Pc)$ after applying it to this method.

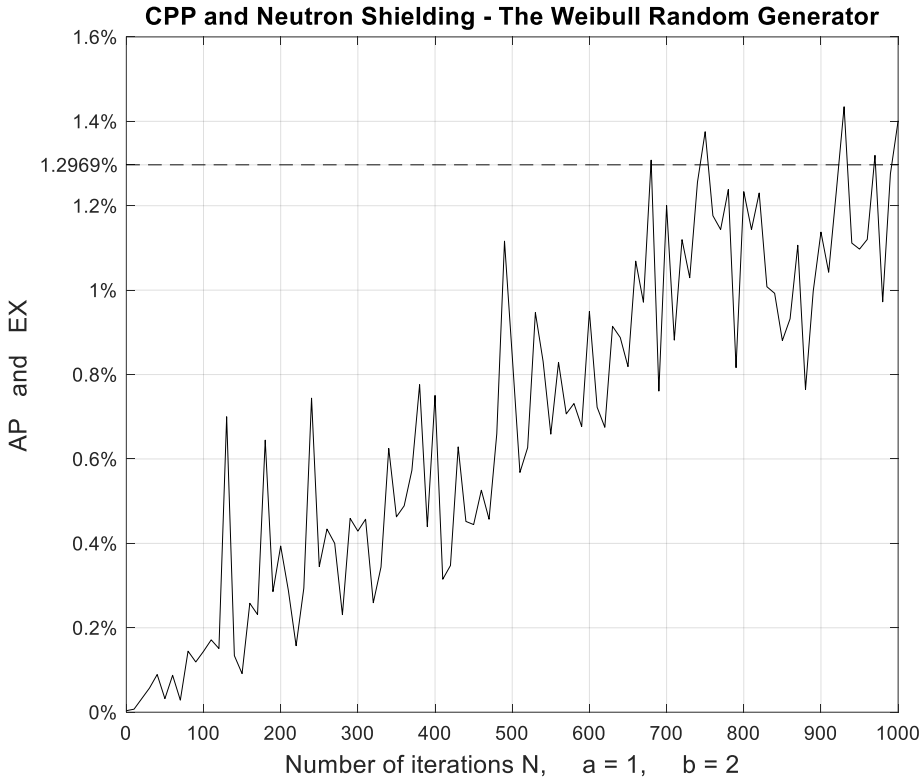


Figure 20- The increasing convergence of the neutron shielding algorithm up to N = 1000 iterations with the Weibull random number generator.

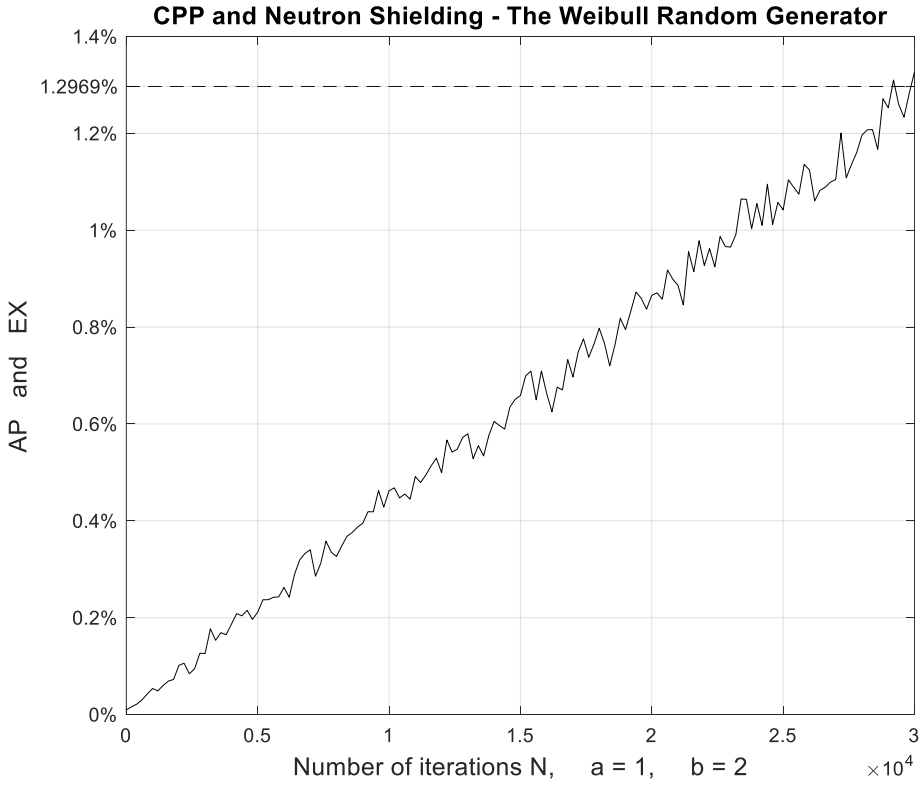


Figure 21- The increasing convergence of the neutron shielding algorithm up to N = 30,000 iterations with the Weibull random number generator.

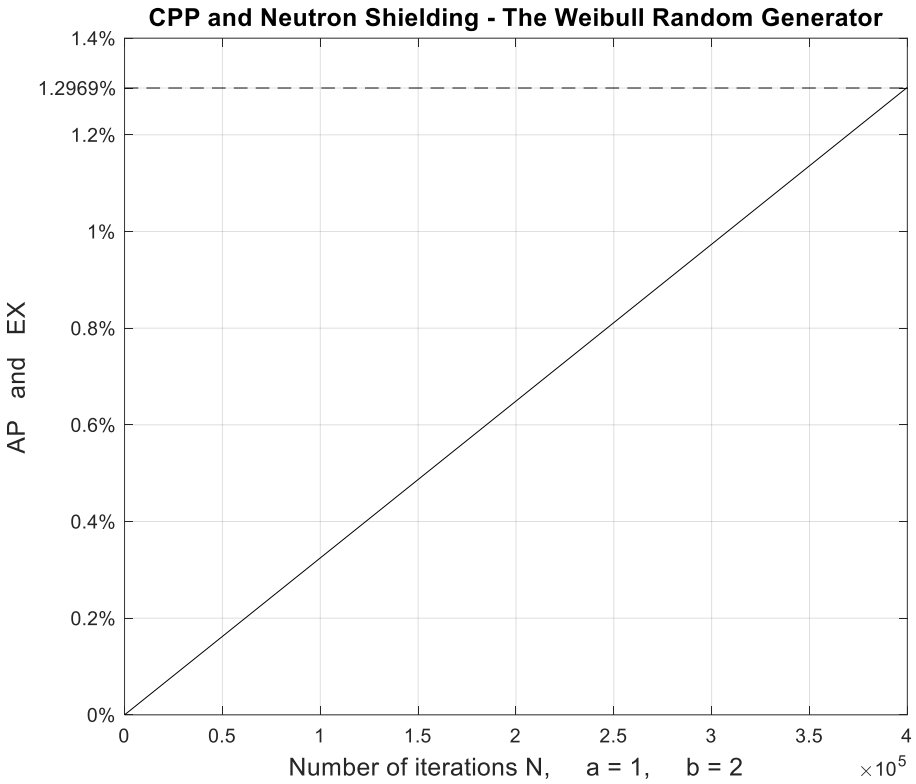


Figure 22- The increasing convergence of the neutron shielding algorithm up to N = 400,000 iterations with the Weibull random number generator.

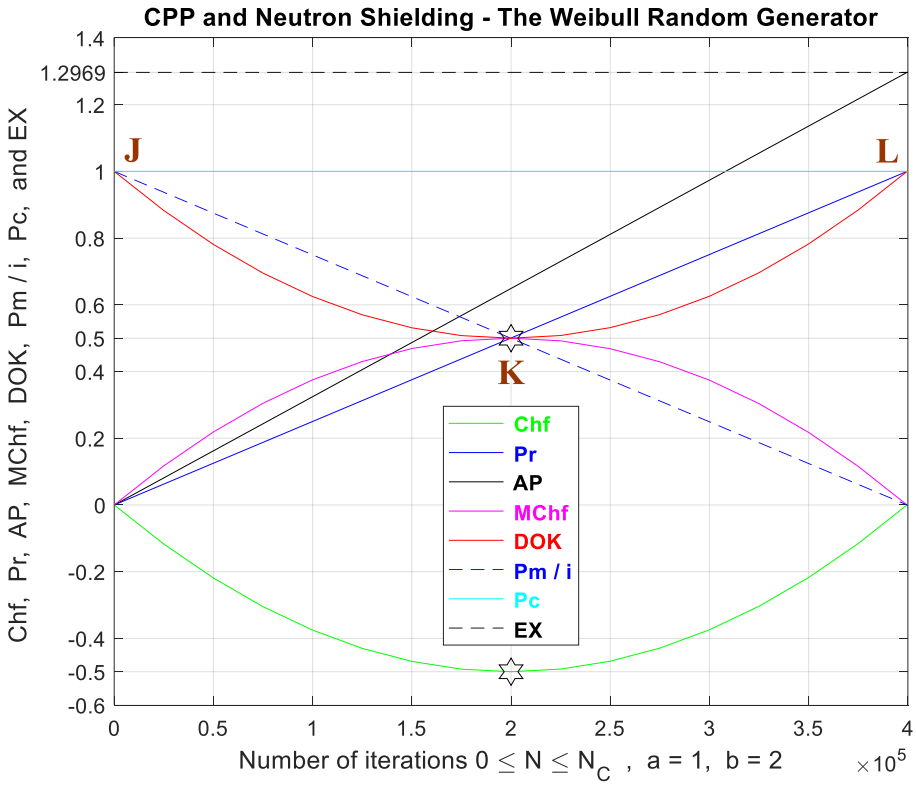


Figure 23- The *CPP* parameters and the neutron shielding algorithm with the Weibull random number generator.

10- Conclusion and Perspectives

In the current research chapter, the original extended Kolmogorov model of eight axioms (*EKA*) was connected and applied to the classical and random neutron shielding numerical technique. Thus, a tight link between the neutron shielding algorithms and the novel paradigm was executed. Accordingly, the model of "Complex Probability" was more expanded beyond the scope of my earlier research studies on this subject.

Also, as it was verified and demonstrated in the original model, when $N = 0$ (before the random simulation beginning) and when $N = N_C$ (when the neutron shielding algorithm converges to the exact result) then the degree of our knowledge (*DOK*) is 1 and the chaotic factor (*Chf* and *MChf*) is 0 since the stochastic effects and fluctuations have either not commenced yet or they have terminated their task on the random experiment. During the course of the nondeterministic experiment ($N > 0$) we have: $0.5 \leq DOK < 1$, $-0.5 \leq Chf < 0$, and $0 < MChf \leq 0.5$. We notice that during this entire process we have continually and incessantly $Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$, that means that the simulation which looked to be random and nondeterministic in the set $\mathcal{R}$ is now deterministic and certain in the set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after adding the contributions of $\mathcal{M}$ to the experiment happening in $\mathcal{R}$ and thus after removing and subtracting the chaotic factor from the degree of our knowledge. Additionally, the probabilities of convergence and divergence of the random neutron shielding procedure that correspond to each iteration cycle N have been determined in the three sets of probabilities which are $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$ by P_r, P_m, and Pc respectively. Subsequently, at each instance of N, the novel neutron shielding technique and *CPP* parameters EX, AP, P_r, P_m, P_m / i, *DOK*, *Chf*, *MChf*, *Pc*, and *Z* are perfectly and surely predicted in the set of complex probabilities $\mathcal{C}$ with *Pc* kept as equal to 1 continuously and forever.

Furthermore, using all these shown simulations and obtained graphs all over the entire research chapter, we can visualize and quantify both the certain knowledge (expressed by *DOK* and *Pc*) and the system chaos and stochastic influences and effects (expressed by *Chf* and *MChf*) of the neutron shielding algorithms. This is definitely very wonderful, fruitful, and fascinating and demonstrates once again the advantages of extending the five axioms of probability of Kolmogorov and thus the benefits and novelty of this original theory in applied mathematics and prognostics that can be called verily:

"The Complex Probability Paradigm".

Moreover, it is important to state here that four essential and very well-known random numbers generators were taken into consideration in the current chapter which are the uniform, Gaussian, Poisson and Weibull random numbers generators, knowing that the original *CPP* model can be applied to any generator of random numbers that exist in literature. This will lead certainly to analogous results and conclusions and will confirm without any doubt the success of my innovative theory.

As a prospective and future challenges and research, we intend to more develop the novel conceived prognostic paradigm and to apply it to a diverse set of nondeterministic events like for other stochastic phenomena as in the classical theory of probability and in stochastic processes. Additionally, we will implement *CPP* to the field of prognostic in engineering and also to other

scientific problems which have huge consequences when applied to economics, to chemistry, to physics, to pure and applied mathematics.

Data Availability

The data used to support the findings of this study are available from the author upon request.

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