

# Global Set Theory(Edition 2)

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# Chapter 1

## Introduction

The law of nature is described in the language of classical logic based on the assumption that the truth value of propositions is either true (1) or false (0). The true (1) and the false (0) are expressions of the degree of truth, and called truth values. Truth values 1 and 0 form a complete Boolean lattice  $\mathbf{2}$  with the order relation  $\leq$  and operations  $\vee$ ,  $\wedge$ ,  $\neg$  :

$$\mathbf{2} = \langle \{1, 0\} : \wedge(\text{and}), \vee : (\text{or}), \neg(\text{not}) \rangle$$

In a sense, the algebra of the 2-valued logic  $2 = \{1, 0\}$  represents the logocentric human interest. A logical counterpart of complete Boolean lattice is called a **classical logic**.

Intuitionistic logic drops the law of excluded middle of classical logic, while quantum logic replaces the full distributive law of classical logic by a weaker distributivity. These logics are two essential non-classical logical systems.

The usual set theory is formulated on basis of classical logic with the axioms of set theory like that of ZFC. The whole mathematics can be formulated in the set theory.

Propositions of set theory are build up starting from formulas of the form " $a \in b$ " ( $a$  is an element of  $b$ ), and then applying recursively the logical connectives (occasionally replacing constants with variables) :

$$\wedge(\text{and}), \vee(\text{or}), \neg(\text{not}), \forall(\text{for all}), \exists(\text{exists}), \supset (\text{implies})$$

Classical set theory is developed in the frame of the classical logic with axioms which claim basic properties (like Extendibility) and existence of each specific sets. The whole sets form the **universe** of set theory. The universe of classical set theory may be reinterpreted as the class model constructed inductively starting from the empty set  $\emptyset$  ( $= V_0^{\mathbf{2}}$ ), and then extending recursively the part of the universe obtained so far by adding ("classical" characteristic functions of) elements of its power set.

$$\begin{aligned} V_{\alpha}^{\mathbf{2}} &= \{u \mid \exists \beta \in \alpha (u \subset V_{\beta}^{\mathbf{2}})\}, \\ V^{\mathbf{2}} &= \bigcup_{\alpha \in \text{On}} V_{\alpha}^{\mathbf{2}}, \end{aligned}$$

where

$$u \subset V_{\beta}^{\mathbf{2}} \stackrel{\text{def}}{\iff} u : V_{\beta}^{\mathbf{2}} \rightarrow \mathbf{2}.$$

$V^{\mathbf{2}}$  is simply written as  $V$ .

Mathematics, the language of science, is developed in the classical set theory.

G.Gentzen formalized a system of the classical logic, which is called LK (cf.[3]). **Formulas** which represent propositions of set theory are constructed from atomic formulas of the forms :

$$u = v \quad \text{or} \quad u \in v$$

by using logical operations :

$$\wedge(\text{and}), \vee(\text{or}), \neg(\text{not}), \forall(\text{for all}), \exists(\text{exists}), \supset(\text{implies}),$$

where the implication  $\supset$  denotes the lattice operation satisfying :

$$a \leq (b \supset c) \iff a \wedge b \leq c.$$

The lattice order  $\leq$  for classical logic is representable as combination of  $\neg$  and  $\vee$  :

$$\varphi \supset \psi \stackrel{\text{def}}{\iff} \neg \varphi \vee \psi \quad \text{since} \quad a \supset b = \begin{cases} 1 & \text{if } a \leq b \iff \neg a \vee b = 1 \\ 0 & \text{otherwise} . \end{cases}$$

Hence, the implication  $\supset$  is dispensable for the classical logic.

However, the quantum physics does not follow all the rules of the classical logic. Quantum logic is not compatible with the distributive law of the classical logic. In particular, the truth values set of quantum logic do not form a Boolean lattice. The algebraic counterpart of quantum logic is an orthomodular lattice. The axiom of orthomodular lattice is obtained from that of Boolean lattice by replacing the distributive law by the weaker law :

$$a \leq b \implies b = a \vee (b \wedge \neg a).$$

Let  $Q$  be an orthomodular lattice which is the set of truth values of an instance of the quantum logic. Then  $Q$ -valued universe  $V^Q$  constructed in  $V$  is the universe of the quantum set theory, in which quantum theory may be described.

For formalization of non-classical logics, we introduce a new logical operator  $\overset{\square}{\rightarrow}$  called **basic implication**, representing :

$$a \overset{\square}{\rightarrow} b = \begin{cases} 1 & a \leq b \\ 0 & \text{otherwise,} \end{cases}$$

which should replace the classical implication  $\supset$ . The basic implication is a logical operation representing the lattice order :

$$a \text{ implies } b \iff a \leq b \iff (a \overset{\square}{\rightarrow} b) = 1.$$

A logic with the basic implication is called a **global logic**. **Global set theory** is based on the global logic.

**Lattice valued logic**, which is a logical counterpart of complete lattice, is formalized as a global logic. Logical operations of lattice valued logic are :

$$\vee, \wedge, \exists, \forall \text{ and the basic implication } \overset{\square}{\rightarrow}.$$

If we introduce a logical operator  $\square$  representing :

$$\square a \stackrel{\text{def}}{\iff} \begin{cases} 1 (= a \vee a^\perp) & \text{if } a = 1 \\ 0 (= 1^\perp) & \text{otherwise} \end{cases}$$

then the basic implication  $\overset{\square}{\rightarrow}$  is replaced by using  $\square$  and  $\supset$  :

$$a \overset{\square}{\rightarrow} b \overset{\text{def}}{\iff} \square(a \supset b) = \begin{cases} 1 & (a \supset b) = 1 \\ 0 & \text{otherwise} \end{cases}$$

In §6, we construct universes of various truth value sets in the universe  $V$ . Then, the metatheory of global set theory is subsumed in the global set theory itself.

Axioms GZFC of global set theory is obtained from ZFC by rephrasing in the global logic. Axioms GNBG of global von-Neumann-Bernays-Gödel set theory is a conservative extension of GZFC with the notion of class. Global von-Neumann-Bernays-Gödel set theory is developed by the global logic based on axioms GNBG.

In §7, we deal with quantum set theory developed on the axioms GNBG by **quantum logic**, which is a counterpart of complete orthomodular lattice. Birkhoff and von Neumann proposed quantum logic represented by the complete orthomodular lattice in [2].

Projections on a Hilbert space  $\mathcal{H}$  form an atomic complete orthomodular lattice with symmetry transformations, which is denoted by  $Q(\mathcal{H})$ . An atomic complete orthomodular lattice is called a **propositional system**.

The logical operators of quantum logic are :

$$\vee, \wedge, \exists, \forall, \perp, \overset{\square}{\rightarrow}.$$

A  $Q(\mathcal{H})$ -valued universe  $V^{Q(\mathcal{H})}$ , which is a universe of a quantum set theory, is constructed in  $V$ . Mathematics described in the quantum universe  $V^{Q(\mathcal{H})}$  can be reconsidered from the view point of the 2-valued universe  $V$ : The quantum universe  $V^{Q(\mathcal{H})}$  is an inner universe constructed in  $V$ . The automorphisms  $\mathcal{U}$  on  $\mathcal{H}$  induce the symmetry transformation on  $Q(\mathcal{H})$ , and hence on  $V^{Q(\mathcal{H})}$ . Set theory on the  $Q(\mathcal{H})$ -valued universe  $V^{Q(\mathcal{H})}$  is called a **Hilbert quantum set theory**.

A maximal compatible subset of  $Q(\mathcal{H})$  is a complete Boolean lattice, say  $B$ . Then

$$Q(\mathcal{H}) = \bigcup_{\sigma \in \mathcal{U}} \sigma(B).$$

$Q(\mathcal{H})$  has a sheaf structure  $\text{Sh}_{\mathcal{U}}B$  of Boolean lattice over  $\mathcal{U}$ , and the orthomodular-lattice-valued universe  $V^{Q(\mathcal{H})}$  has the sheaf structure  $\text{Sh}_{\mathcal{U}}V^B$  of Boolean valued universe over  $\mathcal{U}$ .

Real numbers are defined as Dedekind cuts in a Boolean valued sub-universe  $V^B$ , and represented by self-adjoint operators on  $\mathcal{H}$ . An ‘observable’ in quantum theory is represented by a real number in the quantum universe  $V^{Q(\mathcal{H})}$ . Therefore, in the sheaf representation of the universe  $V^{Q(\mathcal{H})}$ , ‘observables’ are represented by self-adjoint operators moving on  $\mathcal{U}$ .

In §6 (6.4.2), we define **check set**. Check set is a classical member in  $V^{Q(\mathcal{H})}$ . Each set  $x$  in  $V$  corresponds to a check set  $\check{x} \in V^{Q(\mathcal{H})}$ . Let  $W$  be the class of check sets  $\check{x}$  corresponding  $x \in V$ . Then  $W$  is a universe of classical set theory in  $V^{Q(\mathcal{H})}$ .

$$W = \check{V} \subset V^{Q(\mathcal{H})}$$

Since  $W$  in  $V^{Q(\mathcal{H})}$  is isomorphic to  $V$ ,  $W$  includes various universes as if  $V$  includes various universes. It means that various universes are nested, i.e. each universe is an inner universe of the others.

Since universe  $V$  is constructed from empty set  $\emptyset$  and empty set is a subset of every set, universe  $V$  is constructed everywhere in  $V$ .

This mysterious construction is similar to the construction of the universe of Kegon Sutra (Avatamska Sutra).

Logical science, which is founded on the base of global classical logic, comprehends extensive field including quantum physics. Thus, we can observe and describe various inner universes from the point of view of the 2-valued universe  $V$  of classical set theory.

However, it seems to cover still only a bounded aspect of nature. We might be able to perceive more in assimilation with the nature. The whole nature seems to be far beyond the scope of logic.

## Chapter 2

### Naive set theory

We use some symbols as abbreviations of words of informal language.

$A \implies B$  : “ if  $A$  then  $B$  ”

$A \iff B$  : “  $A$  if and only if  $B$  ”

$A \overset{\text{def}}{\iff} B$  : “  $A$  is defined as  $B$  ”

$a \overset{\text{def}}{=} b$  : “  $a$  is defined as  $b$  ”

$\forall x A(x)$  : “ for all  $x$ ,  $A(x)$  ”

$\exists x A(x)$  : “ there exists  $x$  such that  $A(x)$  ”

Propositions in a set theory is represented as formulas constructed from atomic formulas of the form:

$$u \in v \text{ (} u \text{ is a member of } v \text{)} \quad \text{or} \quad u = v \text{ (} u \text{ is equal to } v \text{)}$$

by operations  $\wedge, \vee, \neg, \forall, \exists$  ;

$$\varphi \wedge \psi \text{ (} \varphi \text{ and } \psi \text{),} \quad \varphi \vee \psi \text{ (} \varphi \text{ or } \psi \text{),} \quad \neg \varphi \text{ (not } \varphi \text{),}$$

$$\forall x \varphi(x) \text{ (for all } x \text{ } \varphi(x) \text{),} \quad \exists x \varphi(x) \text{ (} \varphi(x) \text{ for some } x \text{).}$$

We suppose that atomic formulas  $u \in v$ ,  $u = v$  are either true or false. Then every formula is either true(1) or false(0). 1 and 0 are called **truth**

**values.** The set  $\{1, 0\}$  form a complete Boolean algebra, denoted by  $\mathbf{2}$ . Logical operators  $\wedge, \vee, \neg, \forall, \exists$  are interpreted on the complete Boolean algebra as the corresponding Boolean operators.

Truth value of a formula  $\varphi$  is denoted by  $\llbracket \varphi \rrbracket$ .

$$\llbracket \varphi \rrbracket = 1 \text{ (true) or } 0 \text{ (false)}$$

The logical structure of a classical set theory is represented by the algebraic structure of Boolean algebra  $\mathbf{2}$ .

## Sets and formulas

Objects of set theory are sets, and it is indicated by the elements which belong to the set, or by a condition that the elements satisfy :

$$\{a_1, a_2, \dots a_n\} \quad \text{or} \quad \{x \mid \varphi(x)\}$$

A set  $u$  is also expressed by the characteristic function  $\chi_u : \mathcal{D}u \rightarrow \mathbf{2}$ , where  $\mathcal{D}u$  denotes the domain:

$$\chi_u(x) = \begin{cases} 1, & x \in u \\ 0, & x \notin u \end{cases} \quad \text{for } x \in \mathcal{D}u$$

Two sets  $u$  and  $v$  are **equal** if and only if they have the same members :

$$u = v \stackrel{\text{def}}{=} (x \in u \iff x \in v \quad \text{for every } x.)$$

There is only one set which has no elements at all. This set is called the **empty set**, and denoted by the symbol  $\emptyset$ .

We say that  $u$  is a **subset** of  $v$ , in symbols  $u \subset v$ , if every element of  $u$  belongs to  $v$ .

Given sets  $u$  and  $v$ , one can perform some basic operations with them yielding the new sets:

- (1) the set  $u \cup v$  called the **union** of  $u$  and  $v$ , whose elements are the elements of  $u$  or  $v$ .

$$u \cup v = \{x \mid x \in u \vee x \in v\}$$

- (2) the set  $u \cap v$  called the **intersection** of  $u$  and  $v$ , whose elements are the elements common to  $u$  and  $v$ .

$$u \cap v = \{x \mid x \in u \wedge x \in v\}$$

- (3) the set  $u - v$  called the **difference** of  $u$  and  $v$ , whose elements are those elements of  $u$  that are not elements of  $v$ .

$$u - v = \{x \mid x \in u \wedge \neg(x \in v)\}$$

Thus, logical operations  $\vee$ ,  $\wedge$  and  $\neg$  correspond to the set theoretical operations  $\cup$ ,  $\cap$  and complement.

The set theoretical operations satisfy the following properties:

- (1) Associativity:

$$u \cup (v \cup w) = (u \cup v) \cup w, \quad u \cap (v \cap w) = (u \cap v) \cap w$$

- (2) Commutativity:

$$u \cup v = v \cup u, \quad u \cap v = v \cap u$$

- (3) Distributivity:

$$u \cup (v \cap w) = (u \cup v) \cap (u \cup w), \quad u \cap (v \cup w) = (u \cap v) \cup (u \cap w)$$

- (4) Idempotency:

$$u \cup u = u, \quad u \cap u = u$$

- (5) Empty set:

$$u \cup \emptyset = u, \quad u \cap \emptyset = \emptyset, \quad u - u = \emptyset$$

- (6) If  $u \subset v$ , then

$$u \cup v = u \cup (v - u) = v, \quad u \cap v = u.$$

If  $u = v$ , then  $\{u, v\}$  is denoted by  $\{u\}$ .

We define **ordered pair**  $(u, v)$  as the set  $\{\{u\}, \{u, v\}\}$ . Then

$$(u_1, v_1) = (u_2, v_2) \quad \text{if and only if} \quad u_1 = u_2 \text{ and } v_1 = v_2.$$

## Relations

A **binary relation**  $R$  on a set  $X$  is a set of ordered pairs of elements of  $X$ .

$$R \subset X \times X = \{(u, v) \mid u, v \in X\}$$

A binary relation  $R$  on a set  $X$  is called **reflexive** if  $(x, x) \in R$  for every  $x \in X$ . It is called **symmetric** if  $(v, u) \in R$  whenever  $(u, v) \in R$ . And it is called **transitive** if  $(u, w) \in R$  whenever  $(u, v) \in R$  and  $(v, w) \in R$ . A relation that is reflexive, symmetric and transitive is called an **equivalence relation**. If  $R$  is an equivalence relation on a set  $X$  and  $(u, v) \in R$ , then we say that  $u$  and  $v$  are  **$R$ -equivalent**. For  $u \in X$ , the set of all elements of  $X$  that are  $R$ -equivalent to  $u$  is called  **$R$ -equivalence class of  $u$** . The set of all  $R$ -equivalence classes is called the **quotient set** and denoted by  $X/R$ .

For a binary relation  $R$ , one usually write  $aRb$  instead of  $(a, b) \in R$ .

## Functions

A **function** on a set  $X$  is a binary relation  $F$  on  $X$  such that for every  $a \in X$  there exists exactly one pair  $(a, b) \in F$ . Then the element  $b$  is called the **value** of  $F$  at  $a$ , and denoted by  $F(a)$ . The set  $\{x \in X \mid \exists y((x, y) \in F)\}$  is called the **domain** of  $F$ . The notation  $F : X \rightarrow Y$  indicates that  $F$  is a function with domain  $X$  and values in the set  $Y$ .

A function  $F : X \rightarrow Y$  is said to be **one-to-one** if  $a \neq b$  implies  $F(a) \neq F(b)$  for elements  $a, b$  of  $X$ . And  $F$  is said to be **onto** if for every  $b \in Y$  there is some  $a \in X$  such that  $F(a) = b$ . Finally,  $F$  is said to be **bijection** if it is one-to-one and onto.

Given functions  $F : X \rightarrow Y$  and  $G : Z \rightarrow W$ , the **composition** of  $F$  and  $G$ , written  $G \circ F$ , is the function  $G \circ F : X \rightarrow W$  whose elements are all pairs  $(x, G(F(x)))$ , where  $x \in X$ .

## Ordinals

A binary relation  $R$  on a set  $X$  is called **antisymmetric** if  $a = b$  whenever  $R(a, b)$  and  $R(b, a)$ . A relation  $R$  on a set  $X$  that is reflexive, antisymmetric, and transitive, is called a **partial order**. If we remove from  $R$  all pair  $(a, a)$  for every  $a \in X$ , then we get **strict partial order**. A partial order on a given

set  $X$  is usually represented by the symbol  $\leq$  and the corresponding strict partial order by  $\lessdot$  or  $<$ . A partial order  $\leq$  on a set  $X$  with the additional property that either  $a \leq b$  or  $b \leq a$ , for all elements  $a$  and  $b$  of  $X$ , is called a **total order, or linear order**.

If  $\leq$  is a linear order on a set  $X$ , then we say that  $a \in X$  is the  **$\leq$ -least element** of  $X$  if there is no  $b \in X$  distinct from  $a$  such that  $b \leq a$ .

A linear order  $\leq$  on a set  $X$  is a **well-order** if every non-empty subset of  $X$  has a  $\leq$ -least element.

A set  $\alpha$  is called an **ordinal** if

- (1)  $\alpha$  is well-ordered with respect to the order  $\leq$  defined by

$$\beta \leq \gamma \stackrel{\text{def}}{\iff} \beta \in \gamma \vee \beta = \gamma,$$

- (2) if  $\beta \in \alpha$  then  $\beta \subset \alpha$ .

The first ordinal number is defined as the empty set  $\emptyset$ . Given an ordinal  $\alpha$ , the next bigger ordinal, called the **successor** of  $\alpha$ , is the set  $\alpha \cup \{\alpha\}$ .

The **finite ordinal numbers** are those obtained by starting with  $\emptyset$  and repeatedly taking the successor.

In the set theory, the **natural numbers** are defined as the finite ordinals. Thus,

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \emptyset \cup \{\emptyset\} = \{\emptyset\} \\ 2 &= 1 \cup \{1\} = \{\emptyset, \{\emptyset\}\} \\ &\vdots \\ n &= \{0, 1, 2, \dots, n-1\} \\ &\vdots \end{aligned}$$

$$\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, \dots, n, \dots\}$$

A set  $u$  is **finite** if there is a one-to-one correspondence between some natural number  $n$  and the elements of  $u$ . A set is **infinite** if it is not finite.

The set of all finite ordinals is an ordinal and denoted by  $\omega$ . Thus,  $\omega$  is the set  $\mathbb{N}$  of all natural numbers.

The **cardinality** of a finite set  $u$  is the unique natural number  $n$  such that there is a bijection  $F : n \rightarrow u$ .

## Universe of set theory

Objects of set theory are sets. The class of whole sets is called a **universe of set theory**.

**Underlying universe** of set theory is defined inductively as follows.

$$\begin{aligned} V_\alpha &= \{u \mid \exists \beta < \alpha (u \subset V_\beta)\}, \\ V &= \bigcup_{\alpha \in On} V_\alpha. \end{aligned}$$

The least  $\alpha$  such that  $u \in V_\alpha$  is called the **rank** of  $u$ .

Truth values of atomic formulas are expressed using  $\llbracket \cdot \rrbracket$ :

$$\llbracket u=v \rrbracket = \begin{cases} 1 & u=v \\ 0 & u \neq v \end{cases} \quad \llbracket u \in v \rrbracket = \begin{cases} 1 & u \in v \\ 0 & u \notin v \end{cases}$$

Logical operators  $\wedge, \vee, \neg, \forall, \exists$  are interpreted as algebraic operators on the Boolean algebra **2**:

$$\begin{aligned} \llbracket \varphi \wedge \psi \rrbracket &= \llbracket \varphi \rrbracket \wedge \llbracket \psi \rrbracket \\ \llbracket \varphi \vee \psi \rrbracket &= \llbracket \varphi \rrbracket \vee \llbracket \psi \rrbracket \\ \llbracket \neg \varphi \rrbracket &= \neg \llbracket \varphi \rrbracket \\ \llbracket \forall x \varphi(x) \rrbracket &= \bigwedge_{x \in V} \llbracket \varphi(x) \rrbracket \\ \llbracket \exists x \varphi(x) \rrbracket &= \bigvee_{x \in V} \llbracket \varphi(x) \rrbracket \end{aligned}$$

Then every formula has truth value 1 or 0 on  $V$ .

# Chapter 3

## Lattice theory

### 3.1 Lattices

In a logical system, a propositions are expressed by formulas. Lattice represents the structure of truth value of formulas.

Lattice is an ordered set such that any two elements has supremum and infimum. The truth value of a formula  $\varphi$  is denoted by  $\llbracket \varphi \rrbracket$ . The order relation  $\leq$  of the lattice represents “implies”. That is,

$$\llbracket \varphi \rrbracket \leq \llbracket \psi \rrbracket \text{ means that “}\varphi \text{ implies } \psi\text{”}.$$

**DEFINITION 3.1.1.** *A set  $\mathcal{L}$  is said to be **ordered** if there is a relation  $\leq$  on  $\mathcal{L}$  satisfying*

**O1**  $x \leq x$  (*reflexive law*)

**O2** If  $x \leq y$  and  $y \leq x$ , then  $x = y$  (*antisymmetric law*)

**O3** If  $x \leq y$  and  $y \leq z$ , then  $x \leq z$  (*transitive law*)

**DEFINITION 3.1.2.** *Let  $\langle \mathcal{L}, \leq \rangle$  be an ordered set and  $A$  be a subset of  $\mathcal{L}$ . An element  $a$  of  $\mathcal{L}$  is a **supremum** of  $A$ , when*

- (1)  $\forall x \in A (x \leq a)$ , and
- (2) if  $\forall x \in A (x \leq y)$  then  $a \leq y$ .

$a \in \mathcal{L}$  is an **infimum** of  $A$ , when

- (1)  $\forall x \in A(x \geq a)$ , and
- (2) if  $\forall x \in A(x \geq y)$  then  $a \geq y$ .

The supremum of  $A$  is denoted by  $\bigvee A$ . The infimum of  $A$  is denoted by  $\bigwedge A$ . If  $A$  is a finite subset  $\{a_1, \dots, a_n\} \subset \mathcal{L}$ , then  $\bigvee A$  and  $\bigwedge A$  are denoted by  $a_1 \vee \dots \vee a_n$  and  $a_1 \wedge \dots \wedge a_n$ , respectively.

**DEFINITION 3.1.3.** An ordered set  $\mathcal{L}$  is called a **lattice** if  $\mathcal{L}$  is closed under operations  $\vee$  and  $\wedge$  :

$$\mathcal{L} \text{ is a lattice} \stackrel{\text{def}}{\iff} \forall a, b \in \mathcal{L} ((a \vee b), (a \wedge b) \in \mathcal{L}).$$

A lattice  $\mathcal{L}$  is called a **complete lattice** if  $\mathcal{L}$  is closed under operations  $\bigvee$  and  $\bigwedge$  :

$$\mathcal{L} \text{ is a complete lattice} \stackrel{\text{def}}{\iff} \forall A \subset \mathcal{L} ((\bigvee A), (\bigwedge A) \in \mathcal{L}).$$

The largest element  $\bigvee \mathcal{L}$  of  $\mathcal{L}$  is denoted by 1, and the smallest element  $\bigwedge \mathcal{L}$  of  $\mathcal{L}$  is denoted by 0.

$$\bigvee \mathcal{L} = 1, \quad \bigwedge \mathcal{L} = 0.$$

We assume that  $1 \neq 0$ .

**Example 3.1.1.** The two element set  $\{1, 0\}$  consisting of 1 and 0 is a complete lattice, where  $0 \leq 1$ . This lattice  $\langle \{1, 0\}, \leq; \wedge, \vee \rangle$  is denoted by **2**.

**THEOREM 3.1.1.** The binary operations  $\wedge$  (meet) and  $\vee$  (join) of lattice  $\langle \mathcal{L}, \leq \rangle$  satisfies the following L1–L3:

$$\mathbf{L1} \quad a \wedge b = b \wedge a, \quad a \vee b = b \vee a \quad (\text{commutative laws})$$

$$\mathbf{L2} \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c, \quad a \vee (b \vee c) = (a \vee b) \vee c \quad (\text{associative laws})$$

$$\mathbf{L3} \quad a \wedge (b \vee a) = a = a \vee (b \wedge a), \quad a \vee (b \wedge a) = a = a \wedge (b \vee a) \quad (\text{absorption laws})$$

**THEOREM 3.1.2.** If a set  $\mathcal{L}$  with binary operations  $\wedge, \vee$  and relation  $=$  satisfies the conditions L1, L2, L3, then  $(\mathcal{L}, \leq, \wedge, \vee)$  is a lattice, where

$$a \leq b \stackrel{\text{def}}{\iff} a = a \wedge b.$$

*Proof.* If  $\langle \mathcal{L}, =, \wedge, \vee \rangle$  satisfies the conditions L1, L2, L3, then define  $\leq$  by

$$a \leq b \stackrel{\text{def}}{\iff} a = a \wedge b.$$

By absorption laws,

$$\begin{aligned} a = a \wedge b &\implies a \vee b = (a \wedge b) \vee b = b, \\ a \vee b = b &\implies a \wedge b = (a \vee b) \wedge b = b. \\ \therefore a \leq b &\iff a = a \wedge b \iff b = a \vee b. \end{aligned}$$

(1)  $\langle \mathcal{L}, \leq \rangle$  is an ordered set, because

**O1:**  $a = a \wedge ((a \wedge a) \vee a) = a \wedge (a \vee (a \wedge a)) = a \wedge a$  by L3 and L1.

Hence,  $\leq$  is reflexive.

**O2:** If  $(a \leq b) \wedge (b \leq a)$ , then  $a = (a \wedge b) = (b \wedge a) = b$ . Therefore,  $\leq$  is antisymmetric.

**O3:** If  $(a \leq b) \wedge (b \leq c)$ , then  $a \leq c$ . That is, transitive.

$$\therefore (a = a \wedge b) \wedge (b = b \wedge c), \quad \therefore a = a \wedge b \wedge c = a \wedge c.$$

(2)  $a \vee b$  is the supremum of  $\{a, b\}$ , because

$$a \leq a \vee b \quad \text{since } a = a \wedge (a \vee b) \text{ by L3.} \quad \text{Similarly, } b \leq a \vee b.$$

Assume  $a \leq c$  and  $b \leq c$ . By using L3 again,

$$a = a \wedge c \implies a \vee c = (a \wedge c) \vee c = c.$$

$$c = a \vee c \implies a \wedge c = a \wedge (a \vee c) = a.$$

Hence,  $a = (a \wedge c) \iff c = (a \vee c)$ . Similarly,  $b = (b \wedge c) \iff c = (b \vee c)$ .

Therefore,

$$c = c \vee c = (a \vee c) \vee (b \vee c) = (a \vee b) \vee c. \quad \therefore a \vee b \leq c.$$

(3) Similarly,  $a \wedge b$  is the infimum of  $\{a, b\}$ .

□

**DEFINITION 3.1.4.** A lattice  $\mathcal{L}$  is said to be **distributive** if the following distributive law is satisfied.

**Distributive law:** If  $a \in \mathcal{L}$ ,  $\{b_i\}_i \subset \mathcal{L}$ ,  $\bigvee_i b_i \in \mathcal{L}$  and  $\bigvee_i (a \wedge b_i) \in \mathcal{L}$ , then

$$a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i).$$

*Epecially,  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$  for elements  $a, b, c$  of distributive lattice.*

**THEOREM 3.1.3.** The following two conditions (1) and (2) of a lattice  $\mathcal{L}$  are equivalent.

(1)  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$  for elements  $a, b, c$  of  $\mathcal{L}$ .

(2)  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$  for elements  $a, b, c$  of  $\mathcal{L}$ .

*Proof.* If  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$  for  $a, b, c \in \mathcal{L}$ , then

$$\begin{aligned} (a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) \\ &= a \vee (a \wedge c) \vee (b \wedge c) \\ &= a \vee (b \wedge c) \end{aligned}$$

If  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$  for  $a, b, c \in \mathcal{L}$ , then

$$\begin{aligned} (a \wedge b) \vee (a \wedge c) &= ((a \wedge b) \vee a) \wedge ((a \wedge b) \vee c) \\ &= a \wedge (a \vee c) \wedge (b \vee c) \\ &= a \wedge (b \vee c) \end{aligned}$$

□

**DEFINITION 3.1.5.** If  $a, b$  are elements of a lattice  $\mathcal{L}$  and there exists the greatest element  $c$  of  $\mathcal{L}$  such that  $a \wedge c \leq b$ , then  $c$  is called a **pseudo-complement of  $a$  relative to  $b$** , and denoted by  $a \supset b$  :

$$c \leq (a \supset b) \iff c \wedge a \leq b \quad \text{for every } c \in \mathcal{L} \quad (3.1.1)$$

Classical logic and intuitionistic logic have the corresponding logical operation  $\supset$ .

**DEFINITION 3.1.6.** A lattice  $\mathcal{L}$  is called a **Heyting algebra** if  $\mathcal{L}$  has the largest element 1 and the least element 0, and also if there exists a pseudo-complement  $a \supset b$ , for any elements  $a, b \in \mathcal{L}$ , i.e.

A lattice  $\langle \mathcal{L}, \supset, 0, 1 \rangle$  is a Heyting algebra  $\stackrel{\text{def}}{\iff} \forall a, b \in \mathcal{L} \exists c \in \mathcal{L} (c = (a \supset b))$

**DEFINITION 3.1.7.** A distributive lattice  $\mathcal{L}$  is called **Boolean algebra** or **Boolean lattice** if it is provided with **negation**  $\neg$  satisfying

- (1)  $a \wedge \neg a = 0$ ,
- (2)  $a = \neg \neg a$ , and
- (3)  $a \leq b \implies \neg b \leq \neg a$

.

**THEOREM 3.1.4.** If a complete lattice  $\mathcal{L}$  has a negation and satisfies

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{for } a, b, c \in \mathcal{L},$$

then  $\mathcal{L}$  is a complete Boolean lattice, i.e.  $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$  for  $a \in \mathcal{L}$  and  $\{b_i\}_i \subset \mathcal{L}$ .

*Proof.*  $a \wedge \bigvee_i b_i \geq \bigvee_i (a \wedge b_i)$  is obvious.

If  $a \wedge b_i \leq c$  for all  $i$ , then  $b_i \leq \neg a \vee (a \wedge b_i) \leq \neg a \vee c$  for all  $i$ . It follows that

$$\bigvee_i b_i \leq \neg a \vee c, \quad \therefore a \wedge \bigvee_i b_i \leq a \wedge (\neg a \vee c) \leq c.$$

□

**THEOREM 3.1.5.** Complete Heyting algebra is distributive.

*Proof.* It is obvious that

$$\bigvee_i (a \wedge b_i) \leq a \wedge (\bigvee_i b_i)$$

for elements  $a, b_i$  of the Heyting algebra.

Conversely, if  $a \wedge b_i \leq c$  for all  $i$ , then by (2),  $b_i \leq (a \supset c)$  for all  $i$ , hence  $\bigvee_i b_i \leq (a \supset c)$ . Therefore, by (2) again,  $a \wedge \bigvee_i b_i \leq c$ .

$$\therefore a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i).$$

□

**THEOREM 3.1.6.** *A complete distributive lattice  $\mathcal{L}$  is a Heyting algebra with  $\supset$  defined by*

$$(a \supset b) \stackrel{\text{def}}{=} \bigvee \{x \in \mathcal{L} \mid x \wedge a \leq b\}.$$

*Proof.* If  $c \wedge a \leq b$ , then  $c \in \{x \in \mathcal{L} \mid x \wedge a \leq b\}$ . Hence,  $c \leq (a \supset b)$ .

$$a \wedge (a \supset b) = a \wedge \bigvee \{x \in \mathcal{L} \mid a \wedge x \leq b\} \leq b.$$

□

**DEFINITION 3.1.8.** *If  $\mathcal{L}$  is a Heyting algebra, then operation  $\neg$  is defined by*

$$\neg a \stackrel{\text{def}}{=} (a \supset 0).$$

**THEOREM 3.1.7.** *If  $\mathcal{L}$  is a Heyting algebra, then for  $a, b \in \mathcal{L}$ ,*

$$(1) \quad a \wedge \neg a = 0$$

$$(2) \quad a \leq \neg \neg a$$

$$(3) \quad a \leq b \implies \neg b \leq \neg a$$

**THEOREM 3.1.8.** *An operation  $\supset$  on a Boolean lattice defined by*

$$(a \supset b) \stackrel{\text{def}}{=} \neg a \vee b$$

*is a pseudo-complement of  $a$  relative to  $b$ , so Boolean lattice is an Heyting algebra.*

*Proof.* By distributive law,

$$c \leq (a \supset b) \iff c \wedge a \leq b \quad \text{and} \quad a \wedge (a \supset b) \leq b.$$

□

**Example 3.1.2.** *For a topological space  $X$ , the set  $\mathcal{O}(X)$  of open sets of  $X$  is a complete Heyting algebra with respect to the order of inclusion.*

**Example 3.1.3.** *The power set  $\mathcal{P}(X)$  ( $= \{Y \mid Y \subset X\}$ ) of a set  $X$  is a complete Boolean lattice with respect to the order of inclusion.*

*Epecially, if  $X$  is a singleton  $\{x\}$ , then  $\mathcal{P}(X) = \{X, \emptyset\}$ :*

$$1 = \bigvee \mathcal{P}(X) = \{X\} \quad \text{and} \quad 0 = \bigwedge \mathcal{P}(X) = \emptyset.$$

*The complete Boolean lattice  $\{1, 0\}$  is denoted by  $\mathbf{2}$ .*

**DEFINITION 3.1.9.** Let  $\mathcal{L}$  and  $\mathcal{L}'$  be distributive lattices. A mapping  $f : \mathcal{L} \rightarrow \mathcal{L}'$  is called a **(lattice) homomorphism**, if

$$f(a \wedge b) = f(a) \wedge f(b) \quad \text{and} \quad f(a \vee b) = f(a) \vee f(b) \quad \text{for } a, b \in \mathcal{L}.$$

A lattice homomorphism  $f : \mathcal{L} \rightarrow \mathcal{L}'$  is said to be a **(lattice) isomorphism** if  $f$  is one-to-one and onto mapping.

## 3.2 Stone spaces

The set of all subsets of a set  $X$  is called a power set of  $X$ , and denoted by  $\mathcal{P}(X)$ .  $\mathcal{P}(X)$  is a lattice with respect to order  $\subset$  of inclusion, where the supremum is the union  $\bigcup$  and the infimum is the intersection  $\bigcap$ :

$$\bigcup A \stackrel{\text{def}}{=} \{x \in X \mid \exists a \in A (x \in a)\}, \quad \bigcap A \stackrel{\text{def}}{=} \{x \in X \mid \forall a \in A (x \in a)\}.$$

A subset  $A$  of  $\mathcal{P}(X)$  is called a **set lattice**, if  $A$  is closed under  $\bigcup$  and  $\bigcap$ , i.e.

$$\text{if } B \subset A, \text{ then } \bigcup B \in A \text{ and } \bigcap B \in A.$$

Every lattice which is isomorphic to a set lattice is distributive. The converse Theorem 3.2.3 will be proved later.

**DEFINITION 3.2.1.** Let  $\langle \mathcal{L}, \vee, \wedge \rangle$  be a distributive lattice. A subset  $I$  of  $\mathcal{L}$  is called an **ideal** if

$$\mathbf{I1} : \quad a, b \in I \implies a \vee b \in I.$$

$$\mathbf{I2} : \quad a \in I \text{ and } b \in \mathcal{L} \implies a \wedge b \in I.$$

An ideal  $I$  of the distributive lattice  $\mathcal{L}$  is said to be a **prime ideal** if

$$\mathbf{I3} : \quad a, b \in \mathcal{L} \text{ and } a \wedge b \in I \implies \text{either } a \in I \text{ or } b \in I.$$

**DEFINITION 3.2.2.** An ideal  $I$  of a lattice  $\mathcal{L}$  is said to be **maximal** if

- (1)  $I$  is an ideal of  $\mathcal{L}$  such that  $I \subsetneq \mathcal{L}$ , and
- (2) there is no ideal  $I'$  such that  $I \subsetneq I' \subsetneq \mathcal{L}$ .

**PROPOSITION 3.2.1.** *A maximal ideal of a distributive lattice  $\mathcal{L}$  is a prime ideal.*

*Proof.* Suppose that  $I$  is an ideal which is not prime, i.e. there exist elements  $a, b \in \mathcal{L}$  such that

$$a \wedge b \in I, \quad a \notin I, \quad b \notin I.$$

Then  $I^* = \{x \in \mathcal{L} \mid \exists c \in I(x \leq a \vee c)\}$  is an ideal of  $\mathcal{L}$  such that  $I \subset I^*$ .

$$a \in I^*, \quad (\because) \quad a \leq a \vee c \text{ for } c \in I,$$

$$b \notin I^*, \quad (\because) \text{ if } b \in I^* \text{ then } b = b \wedge (a \vee c) = (a \wedge b) \vee (b \wedge c) \in I \text{ for } c \in I, \\ \text{but } b \notin I.$$

Therefore,

$$I \subsetneq I^* \subsetneq \mathcal{L}, \quad \therefore I \text{ is not maximal.}$$

□

**PROPOSITION 3.2.2.** *For an arbitrary elements  $a, b$  of a distributive lattice  $\mathcal{L}$ , if  $b \not\leq a$ , then there exists a prime ideal  $I$  such that*

$$a \in I \quad \text{and} \quad b \notin I.$$

*Proof.* Let  $\mathcal{I}$  be the set of all ideals of  $\mathcal{L}$ , and  $\mathcal{I}'$  be the set of all ideals  $I$  such that

$$a \in I \quad \text{and} \quad b \notin I \tag{3.2.1}$$

$\mathcal{I}'$  is not empty, because  $\{x \in \mathcal{L} \mid x \leq a\}$  is an ideal to which  $a$  belongs and  $b$  does not belong. If  $\mathcal{J}$  is a nonempty ordered subset of  $\mathcal{I}'$ , then  $(\bigcup \mathcal{J}) \in \mathcal{I}'$ . Therefore, by Zorn's lemma, there exists a maximal ideal in  $\mathcal{I}'$ . Let  $I$  be a maximal ideal in  $\mathcal{I}'$ . It suffices to show that  $I$  is a prime ideal.

Suppose that  $I$  is not prime, i.e. there exist elements  $a_1, a_2 \in \mathcal{L}$  such that

$$a_1 \wedge a_2 \in I, \quad \text{but} \quad a_1 \notin I \quad \text{and} \quad a_2 \notin I.$$

Let  $I_i$  be the ideal generated by  $\{a_i\} \cup I$ ,  $i = 1, 2$ , i.e.

$$I_i = \{x \in \mathcal{L} \mid \exists c \in I(x \leq a_i \vee c)\}.$$

One of the ideals  $I_1$  and  $I_2$  does not contain  $a$ . For,

$$a \in I_1 \wedge a \in I_2 \implies \exists c_1, c_2 \in I(a \leq a_1 \vee c_1 \text{ and } a \leq a_2 \vee c_2).$$

Setting  $c = c_1 \vee c_2$ ,

$$a \in I_1 \wedge a \in I_2 \implies \exists c \in I(a \leq (a_1 \vee c) \wedge (a_2 \vee c))$$

$$a \leq (a_1 \vee c) \wedge (a_2 \vee c) = (a_1 \wedge a_2) \vee (a_1 \wedge c) \vee (a_2 \wedge c) \vee (c \wedge c) \in I.$$

This implies  $a \in I$ , which is impossible. Therefore,  $I$  is a prime ideal satisfying (3.2.1).  $\square$

Let  $X$  be the set of all prime ideals of a distributive lattice  $\mathcal{L}$ , and let

$$h(a) = \{x \in X \mid a \notin x\}, \quad \text{for } a \in \mathcal{L}.$$

$h$  is a mapping  $\mathcal{L} \rightarrow \mathcal{P}(X)$ . Set

$$h(\mathcal{L}) = \{h(a) \subset X \mid a \in \mathcal{L}\}.$$

**THEOREM 3.2.3.** *If  $\mathcal{L}$  is a distributive lattice, then  $h(\mathcal{L})$  is a set lattice and  $h$  is an isomorphism of  $\mathcal{L}$  to  $h(\mathcal{L})$ :*

- (1) *If  $a, b \in \mathcal{L}$  and  $h(a) = h(b)$  then  $a = b$ , that is,  $h$  is one-to-one mapping.*
- (2)  $h(a \wedge b) = h(a) \cap h(b)$ ,
- (3)  $h(a \vee b) = h(a) \cup h(b)$ ,  
for  $a, b \in \mathcal{L}$ , where  $\cap$  and  $\cup$  are set theoretical intersection and union.
- (4)  $a \leq b \iff h(a) \subset h(b)$  for  $a, b \in \mathcal{L}$ .

*Proof.* (1) By Proposition 3.2.2.

- (2) If  $x \in h(a \wedge b)$ , then  $a \wedge b \notin x$ . Then  $a \notin x$  and  $b \notin x$ . Hence,  $x \in h(a) \cap h(b)$ . Conversely, if  $x \in h(a) \cap h(b)$ , then  $a \notin x$  and  $b \notin x$ . Since  $x$  is a prime ideal of  $\mathcal{L}$ ,  $(a \wedge b) \notin x$ , i.e.  $x \in h(a \wedge b)$ .
- (3) If  $x \in h(a \vee b)$ , then  $a \vee b \notin x$ . Then  $a \notin x$  or  $b \notin x$ . Hence,  $x \in h(a)$  or  $x \in h(b)$ , which proves  $x \in h(a) \cup h(b)$ . Conversely, if  $x \in h(a) \cup h(b)$ , then  $a \notin x$  or  $b \notin x$ . Since  $x$  is an ideal of  $\mathcal{L}$ ,  $(a \vee b) \notin x$ , i.e.  $x \in h(a \vee b)$ .
- (4) follows from (1),(2).

$\square$

**COROLLARY 3.2.4.** *For a subset  $\{a_i\}_{i \in I}$  of a distributive lattice  $\mathcal{L}$ , if there exist the supremum  $\bigwedge_i^{\mathcal{L}} a_i$ , and infimum  $\bigvee_i^{\mathcal{L}} a_i$  in  $\mathcal{L}$ , then*

$$h(\bigwedge_i^{\mathcal{L}} a_i) = \bigcap_i h(a_i) = \bigwedge_i^{h(\mathcal{L})} h(a_i) \quad \text{and} \quad h(\bigvee_i^{\mathcal{L}} a_i) = \bigcup_i h(a_i) = \bigvee_i^{h(\mathcal{L})} h(a_i).$$

**DEFINITION 3.2.3.** For a distributive lattice  $\mathcal{L}$ , the set  $X$  of all prime ideals of  $\mathcal{L}$  is called the **Stone space** of  $\mathcal{L}$ , the isomorphism  $h : \mathcal{L} \rightarrow h(\mathcal{L}) \subset \mathcal{P}(X)$ , where  $h(a) = \{x \in X \mid a \notin x\}$ , is called the **Stone isomorphism**, and  $h(\mathcal{L})$  is called the **Stone lattice**.

## 3.3 Completion of Boolean algebras

### 3.3.1 Stone space of Boolean algebra

Stone space of a Boolean algebra  $B = \langle B, \vee, \wedge, \neg \rangle$  is the set of all prime ideals of  $B$ , and the Stone isomorphism  $h$  is an embedding of  $B$  into  $\mathcal{P}(X)$  by Theorem 3.2.3:

$$h : B \rightarrow \mathcal{P}(X), \quad h(a) = \{x \in X \mid a \notin x\},$$

$$h(a \vee b) = h(a) \cup h(b), \quad h(a \wedge b) = h(a) \cap h(b).$$

**PROPOSITION 3.3.1.** If  $B$  is a Boolean algebra, then the Stone lattice  $h(B)$  is a field of subsets of the Stone space  $X$  of  $B$  (Definition 3.2.3) and the Stone isomorphism  $h$  is a Boolean isomorphism of  $B$  onto  $h(B)$ , where

$$h(\neg a) = X - h(a).$$

*Proof.*  $x \in h(\neg a) \Leftrightarrow (\neg a) \notin x \Leftrightarrow a \in x \Leftrightarrow x \notin h(a)$ . □

The Boolean algebra  $\langle h(B), \cup, \cap, {}^c \rangle$  is called the **Stone field of  $B$** , where  $h(a)^c \stackrel{\text{def}}{=} X - h(a)$ .

The Stone space  $X$  is a topological space with open base  $\{h(a) \mid a \in B\}$ , where each  $h(a)$  is closed and open, since  $h(a) = h(\neg \neg a) = X - h(\neg a)$ .

**PROPOSITION 3.3.2.** The Stone space  $X$  of a Boolean algebra  $B$  is a compact totally disconnected Hausdorff space.

*Proof.* If  $x_1, x_2 \in X$  and  $x_1 \neq x_2$ , then there exists an element  $a \in B$  which belongs precisely one of the the sets  $x_1$  and  $x_2$ , say  $a \in x_1$  and  $a \notin x_2$  by Proposition 3.2.2. Consequently  $x_1 \in h(\neg a)$  and  $x_2 \in h(a)$ , and both of  $h(\neg a)$  and  $h(a)$  are open and closed. Thus  $X$  is a totally disconnected Hausdorff space.

To prove the compactness, suppose that  $\{a_t\}_{t \in T} \subset B$  and  $X = \bigcup_{t \in T} h(a_t)$ . It suffices to prove that

$$X = h(a_{t_1}) \cup \cdots \cup h(a_{t_n})$$

for some  $a_{t_1}, \dots, a_{t_n} \in \{a_t\}_{t \in T}$ .

Suppose the contrary, that is, for all  $a_{t_1}, \dots, a_{t_n} \in \{a_t\}_{t \in T}$

$$h(a_{t_1} \vee \cdots \vee a_{t_n}) = h(a_{t_1}) \cup \cdots \cup h(a_{t_n}) \neq X = h(1).$$

Since  $h$  is one-to-one, we infer that

$$a_{t_1} \vee \cdots \vee a_{t_n} \neq 1 \quad \text{for } t_1, \dots, t_n \in T,$$

i.e. that the ideal  $\Delta_0$  generated by all  $a_t$  ( $t \in T$ ) does not contain the unit element 1.  $\Delta_0$  is contained in a maximal ideal  $\Delta$  which is prime, i.e.  $\Delta \in X$ . Since  $a_t \in \Delta$  for  $t \in T$ ,

$$\Delta \not\subset \bigvee_{t \in T} h(a_t) = X$$

which is impossible. □

**PROPOSITION 3.3.3.** *For  $a, a_i \in B$  ( $i \in I$ ), if the equation*

$$a = \bigvee_i a_i \tag{3.3.1}$$

*holds, then*

$$h(a) - \bigcup_i h(a_i)$$

*is closed, nowhere dense subset of the Stone space  $X$ .*

*Proof.* The set  $A = h(a) - \bigcup_i h(a_i)$  is closed since it is the difference of a closed and a open set. Suppose that  $A$  is not nowhere dense, i.e.  $A$  contains a non-empty open set. Then there is an element  $a_0 \in B$  such that

$$h(a_0) \subset A, \quad a_0 \neq 0.$$

Since  $h(a_0) \subset h(a)$ , we have  $a_0 \leq a$  on account of Theorem 3.2.3. Thus  $a \neq a \wedge \neg a_0 \leq a$ . On the other hand,

$$h(a_0) \subset h(a) - h(a_i), \quad \text{i.e. } h(a_i) \subset h(a) - h(a_0) = h(a \wedge \neg a_0).$$

Hence,  $a_i \leq a \wedge \neg a_0$  for every  $i \in I$ . It follows that  $a = \bigvee_i a_i \leq a \wedge \neg a_0$ . This contradicts (3.3.1). □

### 3.3.2 Minimal extensions of Boolean algebras

Every Boolean algebra  $B$  is isomorphic to a subalgebra of a complete Boolean algebra, by Proposition 3.2.3. For instance, the Stone isomorphism maps  $B$  into the complete field of all subsets of the Stone space. This isomorphism does not, in general, preserve infinite joins and infinite meets. The minimal extension defined in this section is an isomorphism  $h^* : B \rightarrow h^*(B)$  of  $B$  into a complete Boolean algebra, which preserves all the infinite joins and meets. A proof is shown as follows ( cf. H.Rasiowa and R.Sikorski [7] ).

**DEFINITION 3.3.1** (MacNeille [5] , Sikorski [8] ). *Let  $X$  be the Stone space of a Boolean algebra  $B$ . A set  $A \subset X$  is said to have the **Baire property** provided there exists an open set  $G$  such that*

$$A - G \quad \text{and} \quad G - A \quad \text{are of the first category.} \quad (3.3.2)$$

**PROPOSITION 3.3.4.** *If  $A, A'$  have the Baire property, then*

$$A \cup A', \quad A \cap A', \quad X - A$$

*have the Baire property.*

*Proof.* Suppose that statement (3.3.2) is satisfied and  $G'$  is an open set such that

$$A' - G' \quad \text{and} \quad G' - A' \quad \text{are of the first category.}$$

Then  $G \cup G'$  is open and

$$(A \cup A') - (G \cup G') \subset (A - G) \cup (A' - G'),$$

$$(G \cup G') - (A \cup A') \subset (G - A) \cup (G' - A').$$

Since the set on the right-hand side are of the first category, so are the sets on the left-hand side.

The complement  $A^c = X - A$  of a set  $A$  with Baire property also has Baire property.

$\therefore$ ) Suppose that (3.3.2) holds, and let  $\overline{G}$  be the closure of  $G$ . The set  $G_0 = \overline{G}^c$  is open and the set  $\overline{G} - G$  is nowhere dense.

$$A^c - G_0 = G_0^c - A = \overline{G} - A \subset (\overline{G} - G) \cup (G - A),$$

$$G_0 - A^c = G_0 \cap A = A \cap \overline{G}^c = A - \overline{G} \subset A - G,$$

which proves that the sets on the left-hand side are of the first category.  $\square$

**DEFINITION 3.3.2.** *Let  $X$  be the Stone space of Boolean algebra  $B$  and  $\mathfrak{B}$  be the class of all sets having the Baire property, which is a field of subsets of space  $X$ .*

*Let  $\Delta$  be the ideal of all sets  $\subset X$  of the first category. The Boolean algebra  $B^* = \mathfrak{B}/\Delta$  is called the **minimal extension**.*

**PROPOSITION 3.3.5.** *The mapping*

$$h^*(a) = |h(a)| \in B^*$$

*is an embedding of Boolean algebra  $B$  into complete Boolean algebra  $B^*$ , where  $h$  is the Stone isomorphism of  $B$  onto the Stone field  $h(B)$  of all both open and closed subset of  $X$ .*

*Proof.*  $h^*$  is a homomorphism of  $B$  into  $B^*$ . If  $h^*(a)$  is zero of  $B^*$ , then  $h(a) \in \Delta$ , i.e. the open set  $h(a)$  is of the first category. Hence  $h(a)$  is empty. Since  $h$  is an isomorphism,  $a$  is the zero element of  $B$ . This proves that  $h^*$  is an isomorphism of  $B$  into  $B^*$ .  $\square$

**THEOREM 3.3.6.** *The minimal extension  $B^*$  of an arbitrary Boolean algebra  $B$  is complete. The canonical isomorphism  $h^*$  preserves all infinite joins and meets, i.e.*

$$\text{if } a = \bigvee_{t \in T}^B a_t, \text{ then } h^*(a) = \bigvee_{t \in T}^{B^*} h^*(a_t), \quad (3.3.3)$$

$$\text{if } a = \bigwedge_{t \in T}^B a_t, \text{ then } h^*(a) = \bigwedge_{t \in T}^{B^*} h^*(a_t). \quad (3.3.4)$$

*Proof.* If (3.3.2) holds, then  $|A| = |G|$ . Therefore every element of  $B^*$  can be represented in the form  $|G|$  where  $G$  is open.

First we shall prove that

$$|\bigcup_{t \in T} G_t| = \bigvee_{t \in T}^{B^*} |G_t| \quad (3.3.5)$$

for every indexed set  $\{G_t\}_{t \in T}$  of open sets of  $X$ .

Let  $G_0$  be the union of all set  $G_t$ . Since  $G_t \subset G_0$ , we have

$$|G_t| \leq |G_0| \quad \text{for every } t \in T.$$

On the other hand, suppose that  $|G| \in B^*$  ( $G$  is open) is an element in  $B^*$  such that

$$|G_t| \leq |G| \quad \text{for every } t \in T.$$

Since  $|G_t - G| = |G_t| - |G| = 0_{B^*}$ , the set  $G_t - \overline{G}$  is open and of the first category, thus it is empty. i.e.

$$G_t \subset \overline{G} \quad \text{for every } t \in T.$$

Hence  $G_0 \subset \overline{G} = G \cup (\overline{G} - G)$  and consequently

$$|G_0| \leq |\overline{G}| = |G| \vee |\overline{G} - G|.$$

Since  $\overline{G} - G$  is nowhere dense,  $|\overline{G} - G| = 0_{B^*}$  and consequently

$$|G_0| \leq |G|$$

which completes the proof of equation (3.3.5). Since every index set of elements of  $B^*$  can be represented in the form  $\{|G_t|\}_{t \in T}$ , where the sets  $G_t$  are open, it follows from (3.3.5) that its join exists, i.e.  $B^*$  is complete.

Suppose that the hypothesis of (3.3.3) holds. We have

$$h(a) = (h(a) - \bigcup_{t \in T} h(a_t)) \cup \bigcup_{t \in T} h(a_t).$$

By Proposition 3.3.3, the element  $|h(a) - \bigcup_{t \in T} h(a_t)|$  is zero of  $B^*$ . Hence

$$h^*(a) = |h(a)| = |\bigcup_{t \in T} h(a_t)| = \bigvee_{t \in T}^{B^*} |h(a_t)| = \bigvee_{t \in T}^{B^*} h^*(a_t)$$

on account of (3.3.5).

(3.3.4) follows from (3.3.3) and De Morgan laws.  $\square$

## 3.4 Completion of Heyting algebra

### 3.4.1 Topological Boolean algebra

**DEFINITION 3.4.1.** *An unary operator  $\circ$  on a Boolean algebra*

$$B = \langle B, \wedge, \vee, \neg, 0, 1 \rangle$$

*is called an **interior operator** if the followings are satisfied for  $a, b \in B$ .*

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$$\mathbf{I1} \quad (a \wedge c)^\circ = a^\circ \wedge c^\circ$$

$$\mathbf{I2} \quad a^\circ \leq a$$

$$\mathbf{I3} \quad a^{\circ\circ} = a^\circ$$

$$\mathbf{I4} \quad 1^\circ = 1$$

$a^\circ$  is called the **interior** of  $a$ . Boolean algebra with interior operation is called a **topological Boolean algebra**. An element  $a$  of a topological Boolean algebra such that  $a = a^\circ$  is called an **open element**. The set of all open elements of an topological Boolean algebra  $B$  is written as  $\mathcal{O}(B)$ .

$$\mathcal{O}(B) \stackrel{\text{def}}{=} \{a \in B \mid a = a^\circ\}$$

**PROPOSITION 3.4.1.** *In a topological Boolean algebra  $B = \langle B, \wedge, \vee, \neg, 0, 1 \rangle$*

$$(1) \quad 0, 1 \in \mathcal{O}(B)$$

$$(2) \quad \text{If } a, b \in \mathcal{O}(B), \text{ then } a \wedge b \in \mathcal{O}(B) \text{ and } a \vee b \in \mathcal{O}(B).$$

### 3.4.2 Embedding of Heyting algebra into a Boolean algebra

In this section, let  $\mathcal{L}$  be a Heyting algebra. Let  $X$  be the Stone space of  $\mathcal{L}$  and  $h$  be the Stone isomorphism:

$$X \stackrel{\text{def}}{=} \{x \subset \mathcal{L} \mid x \text{ is a prime ideal of } \mathcal{L}\},$$

$$h : \mathcal{L} \rightarrow \mathcal{P}(X), \text{ where } h(a) = \{x \in X \mid a \notin x\},$$

$\langle h(\mathcal{L}), \subset, \cap, \cup, \supset \rangle$  is a Heyting algebra which is a sublattice of  $\mathcal{P}(X)$ , where

$$h(a \supset b) = \bigcup \{h(c) \mid c \in \mathcal{L}, a \wedge c \leq b\}.$$

As seen in Theorem 3.2.3, we have

**PROPOSITION 3.4.2.** *For  $a, b, c \in \mathcal{L}$ ,*

$$(1) \quad h(a) = h(b) \iff a = b,$$

- (2)  $h(a) \cap h(b) = h(a \wedge b), \quad h(a) \cup h(b) = h(a \vee b),$
- (3)  $h(0) = \emptyset, \quad h(1) = X,$
- (4)  $h(c) \subset h(a \supset b) \iff h(c) \cap h(a) \subset h(b),$
- (5)  $h(a \supset b) \subset (X - h(a)) \cup h(b)$

Let

$$\begin{aligned} H &\stackrel{\text{def}}{=} h(\mathcal{L}) = \{h(a) \mid a \in \mathcal{L}\} \\ h(a) \wedge_H h(b) &\stackrel{\text{def}}{=} h(a) \cap h(b) \\ h(a) \vee_H h(b) &\stackrel{\text{def}}{=} h(a) \cup h(b) \\ h(a) \supset_H h(b) &\stackrel{\text{def}}{=} h(a \supset b) \\ 0_H &\stackrel{\text{def}}{=} \emptyset, \quad 1_H \stackrel{\text{def}}{=} X \end{aligned}$$

By Proposition 3.4.2,  $\langle H, \wedge_H, \vee_H, \supset_H, 0_H, 1_H \rangle$  is a Heyting algebra which is isomorphic to  $\mathcal{L}$ .

Let  $B$  be the Boolean subalgebra of  $\mathcal{P}(X)$  generated by  $H$ , i.e.

$$B = \{(a_1 \supset_B b_1) \cap \cdots \cap (a_n \supset_B b_n) \mid a_1, \dots, a_n, b_1, \dots, b_n \in H\},$$

where  $(a_i \supset_B b_i) = (X - a_i) \cup b_i$ , and  $0_B = \emptyset, 1_B = X$ . We write  $(a_1 \supset_B b_1) \cap \cdots \cap (a_n \supset_B b_n)$  as  $\bigcap_{i=1}^n (a_i \supset_B b_i)$ .

$$H \subset B \subset \mathcal{P}(X)$$

Now we define an interior  $a^\circ$  of  $a \in B$ .

**LEMMA 3.4.3.** *If  $a_i, b_i \in H$  for  $i = 1, 2, \dots, n$ , then  $\bigcap_{i=1}^n (a_i \supset_H b_i) \in H$ , where  $\bigcap_{i=1}^n (a_i \supset_H b_i) = (a_1 \supset_H b_1) \cap \cdots \cap (a_n \supset_H b_n)$ , and*

$$\bigcap_{i=1}^n (a_i \supset_B b_i) = \bigcap_{i=1}^n (c_i \supset_B d_i) \implies \bigcap_{i=1}^n (a_i \supset_H b_i) = \bigcap_{i=1}^n (c_i \supset_H d_i)$$

for  $a_i, b_i, c_i, d_i \in H$  ( $i = 1, 2, \dots, n$ ).

**DEFINITION 3.4.2** (Funayama [9]). *By Lemma 3.4.2, if element  $a = \bigcap_{i=1}^n (a_i \supset_B b_i) \in B \subset \mathcal{P}(X)$ , then element  $\bigcap_{i=1}^n (a_i \supset_H b_i)$  of  $H$  is uniquely determined. So we define a operation  $\circ$  on  $B$  by*

$$\left( \bigcap_{i=1}^n (a_i \supset_B b_i) \right)^\circ \stackrel{\text{def}}{=} \bigcap_{i=1}^n (a_i \supset_H b_i).$$

$\bigcap_{i=1}^n (a_i \supset_H b_i)$  is the infimum of  $\{(a_i \supset_H b_i) \mid i = 1, \dots, n\}$  in  $H$ , and  $\bigcap_{i=1}^n (a_i \supset_B b_i)$  is the infimum of  $\{(a_i \supset_B b_i) \mid i = 1, \dots, n\}$  in  $B$ .

$$\bigwedge_i^H (a_i \supset_H b_i) = \bigcap_{i=1}^n (a_i \supset_H b_i) \quad \text{and} \quad \bigwedge_i^B (a_i \supset_B b_i) = \bigcap_{i=1}^n (a_i \supset_B b_i).$$

Then we have

**THEOREM 3.4.4.** *If  $\mathcal{L}$  is a Heyting algebra, then there exists a topological Boolean algebra  $\langle B, \circ \rangle$  such that  $\mathcal{L}$  is isomorphic to  $H = \mathcal{O}(B)$ , i.e. For  $a, c \in B$ ,*

**I1**  $(a \cap c)^\circ = a^\circ \cap c^\circ$

**I2**  $a^\circ \leq a$

**I3**  $a \in H \implies a^\circ = a$

**I4**  $1^\circ = 1$

*Proof.* **I1** if  $a = \bigcap_{i=1}^m (a_i \supset_B b_i)$  and  $c = \bigcap_{j=1}^n (c_j \supset_B d_j)$ , then

$$\begin{aligned} (a \cap c)^\circ &= \left( \bigcap_{i=1}^m (a_i \supset_B b_i) \cap \bigcap_{j=1}^n (c_j \supset_B d_j) \right)^\circ \\ &= \left( \bigcap_{i=1}^m (a_i \supset_H b_i) \right) \cap \left( \bigcap_{j=1}^n (c_j \supset_H d_j) \right) \\ &= a^\circ \cap c^\circ \end{aligned}$$

**I2**  $a^\circ = (1 \supset_B a)^\circ = (1 \supset_H a) \leq (1 \supset_B a) = a$

**I3** Since  $a^\circ \in H$ ,  $a^{\circ\circ} = (1 \supset_H a^\circ) = a^\circ$

**I4**  $1^\circ = (1 \supset_B 1)^\circ = (1 \supset_H 1) = 1$

□

Therefore,  $\circ$  is an interior operation,  $B$  is a topological Boolean algebra and  $H = \mathcal{O}(B) \subset B$ .

**LEMMA 3.4.5.** *For  $\{a_i\}_{i \in I} \subset H$ ,*

- (1) *if  $\bigvee_{i \in I}^B a_i$  exists in  $B$ , then  $\bigvee_{i \in I}^H a_i$  exists and  $\bigvee_{i \in I}^H a_i = \bigvee_{i \in I}^B a_i$ .*
- (2) *if  $\bigwedge_{i \in I}^B a_i$  exists in  $B$ , then  $\bigwedge_{i \in I}^H a_i$  exists and  $\bigwedge_{i \in I}^H a_i = (\bigwedge_{i \in I}^B a_i)^\circ$ .*

*Proof.* (1) Assume  $\bigvee_{i \in I}^B a_i$  exists in  $B$ . Then  $a_i \leq (\bigvee_{i \in I}^B a_i)^\circ$  for  $\forall i \in I$ , and for  $c \in H$ ,  $\forall i (a_i \leq c) \implies \bigvee_{i \in I}^B a_i \leq c$ .

$$\therefore \quad \bigvee_{i \in I}^H a_i = \bigvee_{i \in I}^B a_i.$$

(2) Assume  $\bigwedge_{i \in I}^B a_i$  exists in  $B$ . Then  $\bigwedge_{i \in I}^B a_i \leq a_i$  for  $\forall i \in I$ .

If  $c \in H$  and  $c \leq a_i$  for  $\forall i \in I$ , then  $c \leq \bigwedge_{i \in I}^B a_i$ . Since  $c \in H$ ,

$$c \leq (\bigwedge_{i \in I}^B a_i)^\circ.$$

$$\therefore \quad (\bigwedge_{i \in I}^B a_i)^\circ = \bigvee_{i \in I}^H a_i.$$

□

By the Theorem 3.3.6, The minimal extension  $B^*$  of the Boolean algebra  $B$  is complete, and the canonical isomorphism  $h^*$  preserves all infinite joins and meets. That is,

$$B^* = \mathfrak{B}/\Delta, \quad \text{where}$$

$X$  is the Stone space of Boolean algebra  $B$ ;

$\mathfrak{B}$  is the class of all subset of  $X$  having the Baire property, which is a field of subset of the space  $X$ ;

$\Delta$  is the ideal of all sets  $\subset X$  of the first category.

The mapping

$$h^*(a) = |h(a)| \in B^*$$

is an embedding of Boolean algebra  $B$  into complete Boolean algebra  $B^*$ , where  $h$  is the Stone isomorphism of  $B$  onto the Stone field  $h(B)$  of all both open and closed subset of  $X$ .

Then we have

$$\text{if } a, b \in B, \text{ then } a \leq b \iff h^*(a) \leq h^*(b), \quad (3.4.1)$$

$$\text{if } a = \bigvee_{t \in T}^B a_t, \text{ then } h^*(a) = \bigvee_{t \in T}^{B^*} h^*(a_t), \quad (3.4.2)$$

$$\text{if } a = \bigwedge_{t \in T}^B a_t, \text{ then } h^*(a) = \bigwedge_{t \in T}^{B^*} h^*(a_t), \quad (3.4.3)$$

**DEFINITION 3.4.3.** For  $\{a_i\}_{i \in I} \in B$ ,

$$\begin{aligned} H^* &= \{\bigvee^{B^*} h^*(a_i^*) \mid a_i \in H\} \\ b^{\circ*} &= \bigvee^{B^*} \{h^*(a) \leq b \mid a \in H\} \quad \text{for } b \in B^* \\ \mathcal{O}(B^*) &= \{b^{\circ*} \mid b \in B^*\} \end{aligned}$$

Then we have

**LEMMA 3.4.6.** (1)  $h^*(b^\circ) = h^*(b)^{\circ*}$  for  $b \in B$

(2)  $(a \cap c)^{\circ*} = a^{\circ*} \cap c^{\circ*}$  for  $a, b \in B$

(3)  $a^{\circ*} \leq a$  for  $b \in B$

(4)  $a \in H \implies a^{\circ*} = a$  for  $b \in B$

(5)  $1^{\circ*} = 1$

As a result, we have the following theorem.

**THEOREM 3.4.7.** Heyting algebra can be embedded into a complete Heyting algebra, i.e. there exist a complete Heyting algebra  $H^*$  and embedding  $h^* : H \rightarrow H^*$ .

## 3.5 Ortholattice

**DEFINITION 3.5.1.** A lattice with operation  $^\perp$  is called an **ortholattice** if Axiom C is satisfied:

**Axiom C**

$$\begin{aligned} \text{(C1)} \quad & a^{\perp\perp} = a, \\ \text{(C2)} \quad & a \vee a^\perp = 1, \quad a \wedge a^\perp = 0, \\ \text{(C3)} \quad & a \leq b \implies b^\perp \leq a^\perp. \end{aligned}$$

### 3.5.1 McNeille's completion of orthlattice

**DEFINITION 3.5.2.** *If  $\alpha \subset Q$ , then  $\alpha^\perp$  is the set of all elements of  $Q$  which is orthogonal to  $\alpha$ .*

$$\alpha^\perp \stackrel{\text{def}}{=} \{\xi \in Q \mid \forall a \in \alpha (\xi \leq a^\perp)\}.$$

**LEMMA 3.5.1.** *Let  $\alpha, \beta \subset Q$ .*

- (1)  $0 \in \alpha^\perp$ ;
- (2) *If  $\alpha \subset \beta$ , then  $\beta^\perp \subset \alpha^\perp$ ;*
- (3)  $\alpha \subset \alpha^{\perp\perp}$  and  $\alpha^\perp = \alpha^{\perp\perp\perp}$ ;
- (4)  $\alpha \cap \alpha^\perp = \{0\}$ ;
- (5)  $(\alpha \cup \alpha^\perp)^{\perp\perp} = Q$ .

**DEFINITION 3.5.3.** *The set of all subsets of  $Q$  such that  $\alpha = \alpha^{\perp\perp}$  is denoted by  $\mathcal{L}^{\perp\perp}(Q)$ .*

$$\mathcal{L}^{\perp\perp}(Q) \stackrel{\text{def}}{=} \{\alpha \subset Q \mid \alpha = \alpha^{\perp\perp}\}.$$

**LEMMA 3.5.2.** *If  $a \in Q$ , then*

- (1)  $\{a\}^{\perp\perp} = \{\xi \in Q \mid \xi \leq a\}$ .
- (2)  $\{a\}^\perp = \{a^\perp\}^{\perp\perp}$ .

*Proof.* Using Lemma 3.5.1,

$$(1) \quad \xi \in \{a\}^{\perp\perp} \iff \forall \eta \in \{a\}^\perp (\xi \leq \eta^\perp), \text{ where } a^\perp \in \{a\}^\perp,$$

$$\begin{aligned} \therefore \quad \xi \in \{a\}^{\perp\perp} &\implies \xi \leq a^{\perp\perp} = a. \\ \xi \leq a \wedge \eta \in \{a\}^\perp &\implies a^\perp \leq \xi^\perp \wedge \eta \leq a^\perp \\ &\implies \xi \leq \eta^\perp \\ \therefore \quad \xi \leq a &\implies \xi \in \{a\}^{\perp\perp} \end{aligned}$$

$$(2) \quad \xi \in \{a\}^\perp \iff \xi \leq a^\perp. \quad \therefore \quad \{a\}^\perp = \{a^\perp\}^{\perp\perp}$$

□

**COROLLARY 3.5.3.**  $Q$  is embedded into  $\mathcal{L}^{\perp\perp}(Q)$  by  $a \mapsto \{a\}^{\perp\perp}$ .

**LEMMA 3.5.4.** For  $\alpha, \beta \in Q$ ,

$$\beta^{\perp\perp} \subset \alpha^{\perp\perp} \iff \forall c \in \mathcal{L} \left( \alpha \subset \{c\}^{\perp\perp} \Rightarrow \beta \subset \{c\}^{\perp\perp} \right).$$

*Proof.*

$$\begin{aligned} \left( \beta^{\perp\perp} \subset \alpha^{\perp\perp} \right) \wedge \left( \alpha \subset \{c\}^{\perp\perp} \right) &\implies \beta \subset \beta^{\perp\perp} \subset \alpha^{\perp\perp} \subset \{c\}^{\perp\perp} \\ \therefore \beta^{\perp\perp} \subset \alpha^{\perp\perp} &\implies \forall c \in \mathcal{L} \left( \alpha \subset \{c\}^{\perp\perp} \implies \beta \subset \{c\}^{\perp\perp} \right) \end{aligned}$$

$$\begin{aligned} \forall c \in \mathcal{L} \left( \alpha \subset \{c\}^{\perp\perp} \implies \beta \subset \{c\}^{\perp\perp} \right) \wedge (x \in \alpha^{\perp}) &\implies \forall y \in \alpha (x \leq y^{\perp}) \\ &\implies \forall y \in \alpha (y \leq x^{\perp}) \\ &\implies \forall z \in \beta (z \leq x^{\perp}) \\ &\implies \forall z \in \beta (x \leq z^{\perp}) \\ &\implies x \in \beta^{\perp} \\ \therefore \forall c \in \mathcal{L} \left( \alpha \subset \{c\}^{\perp\perp} \implies \beta \subset \{c\}^{\perp\perp} \right) &\implies \alpha^{\perp} \subset \beta^{\perp} \\ \therefore \forall c \in \mathcal{L} \left( \alpha \subset \{c\}^{\perp\perp} \implies \beta \subset \{c\}^{\perp\perp} \right) &\implies \beta^{\perp\perp} \subset \alpha^{\perp\perp} \end{aligned}$$

□

**LEMMA 3.5.5.** If  $\{a_i\}_i \in Q$  and  $(\bigvee_i a_i) \in Q$ , then  $\{\bigvee_i a_i\}^{\perp\perp} = (\bigcup_i \{a_i\}^{\perp\perp})^{\perp\perp}$ .

*Proof.*  $\{a_i\}^{\perp\perp} \subset \{\bigvee_i a_i\}^{\perp\perp}$  for all  $i$ . Hence,  $(\bigcup_i \{a_i\}^{\perp\perp})^{\perp\perp} \subset \{\bigvee_i a_i\}^{\perp\perp}$ .

If  $\bigcup_i \{a_i\}^{\perp\perp} \subset \{c\}^{\perp\perp}$ , then  $\bigvee_i a_i \leq c$ . Hence,  $\{\bigvee_i a_i\}^{\perp\perp} \subset \{c\}^{\perp\perp}$ . It follows by Theorem 3.5.4 that

$$(\bigcup_i \{a_i\}^{\perp\perp})^{\perp\perp} = \{\bigvee_i a_i\}^{\perp\perp}.$$

□

It follows that

**THEOREM 3.5.6** (McNeille[5]. cf. Titani-Kodera-Aoyama[14]; ).  $\mathcal{L}^{\perp\perp}(Q)$  is a complete ortholattice, in which  $Q$  is embedded, where

$$\alpha \leq \beta \iff \alpha \subset \beta, \quad \bigvee_i \alpha_i = (\bigcup_i \alpha_i)^{\perp\perp}, \quad \bigwedge_i \alpha_i = \bigcap_i \alpha_i.$$

### 3.6 Orthomodular lattices

**DEFINITION 3.6.1.** *Elements  $a, b$  of an ortholattice  $\mathcal{L}$  are said to be **compatible**, in symbols  $a \circ b$ , if the sublattice generated by  $\{a, a^\perp, b, b^\perp\}$  is distributive.*

Let  $\mathcal{L}$  be an ortholattice. “ $b \in \mathcal{L}$  is compatible with a subset  $A \subset \mathcal{L}$ ”, in symbols  $b \circ A$ , means that “ $b$  is compatible with all elements of  $A$ ”:

$$b \circ A \stackrel{\text{def}}{\iff} \forall a \in A (b \circ a).$$

**DEFINITION 3.6.2.** *An ortholattice  $\mathcal{L}$  is said to be **orthomodular** if :*

**Axiom P**  $a, b \in \mathcal{L}, a \leq b \implies a \circ b$ .

Boolean algebra is an orthomodular lattice where the orthocomplementation  $^\perp$  is the negation  $\neg$ .

**THEOREM 3.6.1** (cf. Piron [6]). *For elements  $a, b$  of an orthomodular lattice, the following conditions are equivalent.*

- (1)  $a, b$  are compatible
- (2)  $(a \wedge b) \vee (a^\perp \wedge b) \vee (a \wedge b^\perp) \vee (a^\perp \wedge b^\perp) = 1$
- (3)  $(a \wedge b) \vee (a^\perp \wedge b) = b$
- (4)  $(a \vee b^\perp) \wedge b = a \wedge b$

**THEOREM 3.6.2** (cf. Piron [6]). *If  $\mathcal{L}$  is an orthomodular lattice, and if  $a \in \mathcal{L}, C \subset \mathcal{L}, \bigvee C \in \mathcal{L}, a \circ C$ , then*

$$\bigvee_{c \in C} (c \wedge a) \in \mathcal{L} \quad \text{and} \quad \bigvee_{c \in C} (a \wedge c) = a \wedge (\bigvee C).$$

*If  $a \in \mathcal{L}, C \subset \mathcal{L}, \bigwedge C \in \mathcal{L}$  and  $a \circ C$ , then*

$$\bigwedge_{c \in C} (c \vee a) \in \mathcal{L} \quad \text{and} \quad \bigwedge_{c \in C} (a \vee c) = a \vee (\bigwedge C).$$

**THEOREM 3.6.3** (cf. Piron [6]). *If  $\mathcal{L}$  is an orthomodular lattice, and if  $a \in \mathcal{L}, C \subset \mathcal{L}, \bigvee C \in \mathcal{L}, a \circ C$ , then  $a \circ \bigvee C$ ; and if  $\bigwedge C \in \mathcal{L}$ , then  $a \circ \bigwedge C$ .*

Clearly we have:

**THEOREM 3.6.4.** *If  $a, b, a_i, b_i$  ( $i \in I$ ) are elements of an orthomodular lattice  $\mathcal{L}$  such that  $\bigvee_{i \in I} a_i, \bigwedge_{i \in I} a_i \in \mathcal{L}$ , then*

$$(\bigvee_{i \in I} a_i)^\perp = \bigwedge_{i \in I} a_i^\perp; \quad (\bigwedge_{i \in I} a_i)^\perp = \bigvee_{i \in I} a_i^\perp.$$

**Example 3.6.1.** *Closed subspaces ( or equivalently projections ) of a Hilbert space  $\mathcal{H}$  form a complete orthomodular lattice with respect to inclusion as the order relation:*

$$Q(\mathcal{H}) \stackrel{\text{def}}{=} \{a \subset \mathcal{H} \mid a \text{ is a closed subspace of } \mathcal{H}\},$$

$$a \leq b \stackrel{\text{def}}{\iff} a \subset b, \quad a^\perp \stackrel{\text{def}}{=} \{x \mid \forall y \in a (x \perp y)\}, \quad \text{for } a, b \in Q(\mathcal{H}).$$

$Q(\mathcal{H})$  is isomorphic to  $\mathcal{Q}(\mathcal{H})$ :

$$\mathcal{Q}(\mathcal{H}) \stackrel{\text{def}}{=} \{p : \mathcal{H} \rightarrow \mathcal{H} \mid p \text{ a projection of } \mathcal{H}\},$$

$$p \leq q \stackrel{\text{def}}{\iff} \mathcal{R}(p) \subset \mathcal{R}(q), \quad p^\perp \stackrel{\text{def}}{=} \mathcal{R}(p)^\perp, \quad \text{for } p, q \in \mathcal{Q}(\mathcal{H}),$$

where  $\mathcal{R}(p)$  denotes the range of projection  $p$ .

## 3.7 Implication and globalization

**DEFINITION 3.7.1.** **Implication** is an operation  $\supset$  on a lattice  $\mathcal{L}$  such that

- (1)  $a \supset b = 1 \iff a \leq b$
- (2)  $c \leq (a \supset b) \iff c \wedge a \leq b$  for every  $c$ .

**DEFINITION 3.7.2** (Takeuti [12]). *Takeuti defined operation  $\rightarrow_{\mathbf{T}}$  on an orthomodular lattice :*

$$(a \rightarrow_{\mathbf{T}} b) \stackrel{\text{def}}{=} a^\perp \vee (a \wedge b)$$

*in order to develop a quantum set theory.*

**THEOREM 3.7.1.** *Operation  $\rightarrow_{\mathbf{T}}$  is an implication on the complete orthomodular lattice.*

*Proof.* Since  $a \downharpoonright a^\perp$  and  $a \downharpoonright (a \wedge b)$ ,

$$a \wedge (a^\perp \vee (a \wedge b)) = (a \wedge a^\perp) \vee (a \wedge b) \leq b$$

$$\therefore \quad a \wedge (a \rightarrow_{\mathbf{T}} b) \leq b,$$

and then

$$a \leq b \iff (a \rightarrow_{\mathbf{T}} b) = 1.$$

Therefore,  $\rightarrow_{\mathbf{T}}$  is an implication. □

**THEOREM 3.7.2.** *In an orthomodular lattice, if  $a \downharpoonright c$ , then*

$$c \leq (a \rightarrow_{\mathbf{T}} b) \iff a \wedge c \leq b \wedge c.$$

*In this sense,  $\rightarrow_{\mathbf{T}}$  is considered as a **local implication**.*

*Proof.* Assume  $a \downharpoonright c$ . Then  $c = (a^\perp \wedge c) \vee (a \wedge c)$ . Since  $a \downharpoonright a^\perp$  and  $a \downharpoonright (a \wedge b)$ ,

$$\begin{aligned} c \leq (a \rightarrow_{\mathbf{T}} b) &\implies a \wedge c \leq a \wedge (a^\perp \vee (a \wedge b)) \\ &\implies a \wedge c \leq (a \wedge a^\perp) \vee (a \wedge b) \leq b \\ a \wedge c \leq b &\implies a \wedge c \leq a \wedge b \\ &\implies c = (a^\perp \wedge c) \vee (a \wedge c) \leq a^\perp \vee (a \wedge b) \end{aligned}$$

□

However, Takeuti's implication  $\rightarrow_{\mathbf{T}}$  is not enough to develop a set theory, because it is not transitive:

$$(a \rightarrow_{\mathbf{T}} b) \wedge (b \rightarrow_{\mathbf{T}} c) \not\leq (a \rightarrow_{\mathbf{T}} c).$$

The transitivity of the corresponding logical implication is indispensable for the development of set theory, since equality axioms of set theory which depend on the transitivity of implication are fundamental.

### 3.7.1 Basic implication and globalization on a lattice

Let  $\mathcal{L}$  be a lattice including  $\mathbf{2} = \{1, 0\}$ , where 1 is the greatest element and 0 is the least element of  $\mathcal{L}$ .

An operation  $\rightarrow$  on a lattice is called the **basic implication** if

$$(a \rightarrow b) = \bigvee \{c \in \mathbf{2} \mid c \wedge a \leq b\} = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise.} \end{cases}$$

$\rightarrow$  is an implication (cf. Definition 3.7.1), and the corresponding negation  $\neg$  is defined by

$$\neg a \stackrel{\text{def}}{=} (a \rightarrow 0).$$

Then we have

**THEOREM 3.7.3.** *For all elements  $a, b$  of a complete lattice  $\mathcal{L}$ ,*

$$\mathbf{I1} : (a \rightarrow b) = 1 \text{ iff } a \leq b$$

$$\mathbf{I2} : a \wedge (a \rightarrow b) \leq b.$$

$$\mathbf{N1} : \neg 0 = 1, \quad \neg 1 = 0$$

$$\mathbf{N2} : a \wedge \neg a = 0$$

$$\mathbf{N3} : a \leq \neg \neg a$$

$$\mathbf{N4} : \neg(a \vee b) = \neg a \wedge \neg b$$

**DEFINITION 3.7.3.** *The formula  $(1 \rightarrow a)$  for  $a \in \mathcal{L}$  is denoted by  $\Box a$ , that is,*

$$\Box a \stackrel{\text{def}}{=} (1 \rightarrow a) = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1. \end{cases}$$

*This operator  $\Box$  is called **globalization**.*

**THEOREM 3.7.4.** *For all elements  $a, b, a_k, b_k, c_k$  ( $k \in K$ ) of  $\mathcal{L}$ ,*

$$\mathbf{G1} : \Box a \leq a$$

$$\mathbf{G2} : \neg a = \Box \neg a$$

$$\mathbf{G3} : \bigwedge_k \Box a_k \leq \Box \bigwedge_k a_k$$

$$\mathbf{G4} : \text{If } \Box a \leq b, \text{ then } \Box a \leq \Box b$$

$$\mathbf{G5} : \Box a \wedge \bigvee_k b_k = \bigvee_k (\Box a \wedge b_k); \quad a \wedge \bigvee_k \Box b_k = \bigvee_k (a \wedge \Box b_k), \text{ if } \bigvee_k b_k \text{ exists}; \\ \Box a \vee \bigwedge_k b_k = \bigwedge_k (\Box a \vee b_k); \quad a \vee \bigwedge_k \Box b_k = \bigwedge_k (a \vee \Box b_k), \text{ if } \bigwedge_k b_k \text{ exists}$$

$$\mathbf{G6} : \Box a \vee \neg \Box a = 1$$

$$\mathbf{G7} : \text{If } a \wedge \Box c \leq b, \text{ then } \neg b \wedge \Box c \leq \neg a.$$

$$\mathbf{G8} : (a \rightarrow b) = \bigvee \{c \in \mathcal{L} \mid c = \Box c, \quad a \wedge c \leq b\}$$

The following theorem follows from I1–I2, N1–N4 and G1–G8.

**THEOREM 3.7.5.** *Let  $a, b \in \mathcal{L}$  and  $\{a_k\}_{k \in K}, \{b_k\}_{k \in K} \subset \mathcal{L}$ . Then*

$$(1) \quad \text{If } a \leq b \text{ then } \Box a \leq \Box b$$

$$(2) \quad \Box(\bigwedge_k a_k) = \bigwedge_k \Box a_k$$

$$(3) \quad \Box a = \Box \Box a$$

$$(4) \quad \bigwedge_k \Box a_k = \Box \bigwedge_k \Box a_k$$

$$(5) \quad \bigvee_k \Box a_k = \Box \bigvee_k \Box a_k$$

$$(6) \quad \Box(a \rightarrow b) = (a \rightarrow b).$$

$$(7) \quad (a \rightarrow b) \leq (\neg b \rightarrow \neg a)$$

$$(8) \quad \text{If } \Box a \wedge b \leq c \text{ then } \Box a \leq (b \rightarrow c)$$

We denote  $\neg \Box \neg$  by  $\Diamond$ . Then we have

**THEOREM 3.7.6.** *Let  $a, b \in \mathcal{L}$  and  $\{a_k\}_{k \in K} \subset \mathcal{L}$ .*

$$(1) \quad a \leq \Diamond a$$

$$(2) \quad \text{If } a \leq \Box b \text{ then } \Diamond a \leq \Box b$$

$$(3) \quad \Diamond \bigvee_k a_k = \bigvee_k \Diamond a_k$$

$$(4) \quad \Diamond(\Box a \wedge b) \leq \Box a \wedge \Diamond b$$

**THEOREM 3.7.7.** *If an implication  $\supset$  is defined on a lattice, then the basic implication  $\rightarrow$  is defined by*

$$(a \rightarrow b) \stackrel{\text{def}}{=} \Box(a \supset b).$$

*Conversely, the globalization  $\Box$  is defined in terms of the basic implication  $\rightarrow$ :*

$$\Box a \stackrel{\text{def}}{=} ((a \rightarrow a) \rightarrow a).$$

Furthermore,  $\Diamond$  is defined by  $\Diamond a \stackrel{\text{def}}{=} (\Box a^\perp)^\perp$ .

Clearly we have:

**THEOREM 3.7.8.** *If  $a, b, a_i, b_i$  are elements of an orthomodular lattice  $\mathcal{L}$ , then*

- (1)  $(\bigvee_i a_i)^\perp = \bigwedge_i a_i^\perp$ ;
- (2)  $(\bigwedge_i a_i)^\perp = \bigvee_i a_i^\perp$ ;
- (3)  $\bigvee_i (\Box a_i) = \Box \bigvee_i \Box a_i$ ;
- (4)  $\bigwedge_i \Box a_i = \Box \bigwedge_i \Box a_i$ ;
- (5)  $(\Box a)^\perp = \Box((\Box a)^\perp)$ ;
- (6)  $a \downarrow \Box b$ .

For any complete lattice, basic implication  $\rightarrow$  is defined by

$$(a \rightarrow b) \stackrel{\text{def}}{=} \bigvee \{x \in \mathbf{2} \mid a \wedge x \leq b\} = \begin{cases} 1, & \text{if } a \leq b, \\ 0, & \text{otherwise,} \end{cases}$$

and  $\neg$ ,  $\Box$ ,  $\Diamond$  are defined by

$$\neg a \stackrel{\text{def}}{=} (a \rightarrow 0) = \bigvee \{x \in \mathbf{2} \mid a \wedge x \leq 0\} = \begin{cases} 1, & \text{if } a = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\Box a \stackrel{\text{def}}{=} ((a \rightarrow a) \rightarrow a) = (1 \rightarrow a) = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\Diamond a \stackrel{\text{def}}{=} \neg \Box \neg a = \begin{cases} 1 & \text{if } a \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Obviously we have

**THEOREM 3.7.9.**  $\Box$  is a modal operator on a lattice, satisfying

- (1)  $\Box a \leq a$ ;
- (2)  $\Box \Box a = \Box a$ ;
- (3)  $a \leq b \Rightarrow \Box a \leq \Box b$ .
- (4)  $\bigvee_i (\Box a_i) = \Box \bigvee_i \Box a_i$ ;
- (5)  $\bigwedge_i \Box a_i = \Box \bigwedge_i \Box a_i$ ;

**THEOREM 3.7.10.** If a lattice  $\mathcal{L}$  is an ortholattice, then

- (1)  $\Box a \vee (\Box a)^\perp = 1$ ,  $\Box a \wedge (\Box a)^\perp = 0$ ;
- (2)  $(\Box a)^\perp = \Box ((\Box a)^\perp)$ ;
- (3)  $a \circlearrowleft \Box b$ , where  $a \circlearrowleft c \stackrel{\text{def}}{\iff} a = (a \wedge c) \vee (a \wedge c^\perp)$ .

**DEFINITION 3.7.4.** An element  $a$  of a lattice is said to be **global** if

$$a = \Box a.$$

**Example 3.7.1.** If  $\mathcal{L}$  is a complete Heyting algebra, and operation  $\rightarrow_I$  is defined by

$$a \rightarrow_I b \stackrel{\text{def}}{=} \bigvee \{c \in \mathcal{L} \mid c \wedge a \leq b\},$$

then  $\rightarrow_I$  is an implication and the basic implication  $\rightarrow$  is defined using  $\Box$  :

$$a \rightarrow b \stackrel{\text{def}}{=} \Box(a \rightarrow_I b).$$

**Example 3.7.2.** If  $\mathcal{L}$  is a complete orthomodular lattice, then the operation  $\rightarrow_{\mathbf{T}}$  defined by  $a \rightarrow_{\mathbf{T}} b = a^{\perp} \vee (a \wedge b)$  is an implication. Thus, a basic implication  $\rightarrow$  and  $\neg$  on  $\mathcal{L}$  are defined

$$(a \rightarrow b) \stackrel{\text{def}}{=} \Box(a \rightarrow_{\mathbf{T}} b), \quad \neg a \stackrel{\text{def}}{=} (a \rightarrow 0).$$

As immediate consequents of the definitions, we have:

**THEOREM 3.7.11.** In a complete orthomodular lattice,

- (1)  $\neg 0 = 1$  ;  $\neg 1 = 0$
- (2)  $a \wedge \neg a = 0$  ;  $a \leq \neg \neg a$
- (3)  $(a \rightarrow b) \leq (\neg b \rightarrow \neg a)$
- (4)  $\neg(a \vee b) = \neg a \wedge \neg b$  ;  $\neg a \vee \neg b \leq \neg(a \wedge b)$
- (5)  $\Box \Box a = \Box a$
- (6) If  $\Box a \leq b$ , then  $\Box a \leq \Box b$
- (7)  $a \leq \Diamond a$
- (8) If  $a \wedge \Box c \leq b$ , then  $\neg b \wedge \Box c \leq \neg a$ .
- (9)  $(a \rightarrow b) = \Box(a \rightarrow_{\mathbf{T}} b)$ .

*Especially, on an orthomodular lattice  $\mathcal{Q}$ ,*

- (1)  $(a \rightarrow b) = \bigvee \{c \in \mathcal{Q} \mid c = \Box c, a \wedge c \leq b\}$
- (2)  $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$  ;  $(a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$
- (3)  $(\Box a)^{\perp} = \neg(\Box a)$
- (4) If  $\Box a \wedge b \leq c$  then  $\Box a \leq (b \rightarrow_{\mathbf{T}} c)$
- (5)  $((\Box a \wedge b) \rightarrow_{\mathbf{T}} c) = (\Box a \rightarrow_{\mathbf{T}} (b \rightarrow_{\mathbf{T}} c))$
- (6)  $\Diamond \bigvee_k a_k = \bigvee_k \Diamond a_k$
- (7)  $\Diamond(\Box a \wedge b) = \Box a \wedge \Diamond b$

### 3.7.2 Sheaf of complete Boolean algebra

**DEFINITION 3.7.5.** Let  $X$  be a topological space and  $\mathcal{O}(X)$  be the set of all open sets of  $X$ . Let each  $U \in \mathcal{O}(X)$  associate with a complete Boolean algebra  $F(U)$ , and for  $U, V \in \mathcal{O}(X)$  such that  $U \subset V$ ,

$$r_{U,V} : F(V) \rightarrow F(U) \text{ be an homomorphism.}$$

Then the pair  $\langle F, r \rangle$  is called a **pre-sheaf of complete Boolean algebra** over  $X$ , if

- (1)  $F(\emptyset) = 0$ ,  $r_{U,U} = 1$  (identity),
- (2) If  $U, V, W \in \mathcal{O}(X)$  and  $U \subset V \subset W$ , then  $r_{U,W} = r_{U,V} \circ r_{V,W}$ .

Pre-sheaf  $\langle F, r \rangle$  is called a **sheaf of complete Boolean algebra** over  $X$ , if the following condition is satisfied.

- (3) If  $U \in \mathcal{O}(X)$ ,  $\{U_i\}_{i \in I} \subset \mathcal{O}(X)$  and  $U = \bigcup_i U_i$  and further if

$$\forall i (f_i \in F(U_i)) \wedge \forall i, j \in I (r_{U_i \cap U_j, U_i}(f_i) = r_{U_i \cap U_j, U_j}(f_j)),$$

then there exists a unique  $f \in F(U)$  such that  $\forall i \in I (r_{U_i, U}(f) = f_i)$ .

**DEFINITION 3.7.6.** Let

- $\langle F, r \rangle$  be a sheaf of complete Boolean algebra over  $X$
- $\{U_i : i \in I\} \subset \mathcal{O}(X)$  be a directed system of neighbourhood of  $x$ , such that

$$i \leq j \implies U_j \subset U_i \text{ and } r_{U_j, U_i}(f(U_i)) = f(U_j)$$

- $f(x)$  be the direct limit:

$$f(x) \stackrel{\text{def}}{=} \varinjlim f(U_i).$$

$$B_x \stackrel{\text{def}}{=} \{f(x) \mid \exists U \in \mathcal{O}(X) (f \in F(U) \wedge x \in U)\}.$$

Then  $B_x$  is a complete Boolean algebra called a **stalk** at  $x$ .

**Example 3.7.3** (Sheaf representation of  $Q(\mathcal{H})$ ).  $Q(\mathcal{H})$  is the complete orthomodular lattice consisting of closed subspaces (or projections) of a Hilbert space  $\mathcal{H}$  ( cf. Example 3.6.1), where the inner product is denoted by  $(\cdot, \cdot)$ . Let

$$\{\vec{e}_j\}_{j \in J} \text{ where } J = \{1, 2, \dots\}$$

be a countable orthonormal basis and  $\mathfrak{e}_j$  be the subspace of  $\mathcal{H}$  spanned by  $\vec{e}_j$ :

$$\mathfrak{e}_j = \{a\vec{e}_j \mid a \in \mathbb{C}\}.$$

For each  $K \subset J$ , the supremum  $\bigvee_{j \in K} \mathfrak{e}_j$  of  $\{\mathfrak{e}_j\}_{j \in K}$  in  $Q(\mathcal{H})$  is the subspace of  $\mathcal{H}$  spanned by  $\{\vec{e}_j\}_{j \in K}$ :

$$\bigvee_{j \in K} \mathfrak{e}_j := \{\sum_{j \in K} a_j \vec{e}_j \mid \{a_j\}_{j \in K} \subset \mathbb{C}\}.$$

A subset  $B$  of  $Q(\mathcal{H})$  defined by

$$B = \{\bigvee_{j \in K} \mathfrak{e}_j \mid K \subset J\}$$

is a sublattice of  $Q(\mathcal{H})$ , which is a complete Boolean sub-algebra isomorphic to the power set  $\mathcal{P}(J)$  of  $J$ .

$$\langle B, \wedge, \vee, {}^\perp \rangle \cong \langle \mathcal{P}(J), \cap, \cup, {}^c \rangle$$

A linear operator  $\sigma : \mathcal{H} \rightarrow \mathcal{H}$  is said to be **unitary** if

$$(\sigma(\vec{x}), \sigma(\vec{y})) = (\vec{x}, \vec{y}), \quad \text{for all } \vec{x}, \vec{y} \in \mathcal{H}.$$

Unitary operator induces an isomorphism  $\sigma : Q(\mathcal{H}) \rightarrow Q(\mathcal{H})$  preserving  $\wedge$ ,  $\vee$  and  ${}^\perp$ . Let  $\mathcal{U}$  be a topological space consisting of all unitary operators on  $\mathcal{H}$ :

$$\mathcal{U} = \{\sigma : \mathcal{H} \rightarrow \mathcal{H} \mid \text{unitary}\},$$

and let  $\mathcal{O}(\mathcal{U})$  be the set of open sets.

For each  $\sigma \in \mathcal{U}$ ,  $\{\sigma(\mathfrak{e}_j)\}_{j \in J}$ , where  $\sigma(\mathfrak{e}_j) = \{\sigma(\vec{x}) \mid \vec{x} \in \mathfrak{e}_j\}$ , is a basis of  $Q(\mathcal{H})$ , and  $\bigvee_{j \in K} \sigma(\mathfrak{e}_j)$  with  $K \subset J$  is an element of  $Q(\mathcal{H})$  spanned by  $\{\sigma(\vec{e}_j)\}_{j \in K}$ . Let

$$\sigma(B) \stackrel{\text{def}}{=} \{\bigvee_{j \in K} \sigma(\mathfrak{e}_j) \mid K \subset J\}.$$

$\sigma(B)$  is a complete Boolean algebra isomorphic to  $B$ .

$\sigma(B)$  is a  $(\bigvee, \perp)$ -sublattice of  $Q(\mathcal{H})$  and

$$Q(\mathcal{H}) = \bigcup_{\sigma \in \mathcal{U}} \sigma(B).$$

For each  $U \in \mathcal{O}(\mathcal{U})$ ,

$$F(U) = \left\{ \bigvee_{j \in K} \sigma(\mathfrak{e}_j) \mid K \subset J, \sigma \in U \right\}$$

Then  $F(U)$  is a Boolean algebra.

For  $U, V \in \mathcal{O}(X)$  such that  $U \subset V$ , let  $r_{U,V}(F(V))$  be the set of restriction of elements of  $F(V)$  on  $U$ .

$$r_{U,V}(F(V)) = F(U) = \left\{ \bigvee_{j \in K} \sigma(\mathfrak{e}_j) \mid K \subset J, \sigma \in U \right\}.$$

Then  $\langle F, r \rangle$  is a sheaf of complete Boolean algebra over  $\mathcal{U}$ .

# Chapter 4

## Classical set theory

### 4.1 Formal system of set theory

Set theory ZFC is an axiomatic set theory known as Zermelo-Freankel axioms with axiom of choice, based on the first-ordered logic.

We first introduce Gentzen's first-order logic, known as classical logic LK, and intuitionistic logic LJ.

#### 4.1.1 Gentzen's formal system of logic

##### Alphabet of LK and LJ

- (1) Individual constants :  $c, c_0, c_1, c_2, \dots$ ,
- (2) Individual free variables :  $a, a_0, a_1, a_2, \dots$ ,
- (3) Individual bound variables :  $x, x_0, x_1, x_2, \dots$ ,
- (4) Predicate constants with  $n$  arguments :  $p^n, p_0^n, p_1^n, p_2^n, \dots$ ,  $(n \geq 0)$ ,
- (5) Logical symbols :  $\supset$  (implies),  $\wedge$  (and),  $\vee$  (or),  $\neg$  (not),  $\forall$  (for all),  $\exists$  (exists),
- (6) Auxiliary symbols :  $(, )$  and commas.

## Terms

Individual constants and free variables are called **terms**, and they are denoted by  $t_1, t_2, \dots$ .

## Formulas

If  $p_i^n$  is a predicate constant with  $n$  argument places and  $t_1, \dots, t_n$  are terms, then  $p_i^n(t_{i_1}, \dots, t_{i_n})$  is called an **atomic formula**. Formulas are constructed from the atomic formulas using logical symbols:

- (1) The atomic formulas are formulas.
- (2) If  $\varphi$  and  $\psi$  are formulas, then  $(\varphi \wedge \psi)$ ,  $(\varphi \vee \psi)$  are formulas.
- (3) If  $\varphi(a)$  is a formula with free variable  $a$ , and  $x$  is a bound variable which does not occur in  $\varphi(a)$ , then  $\forall x\varphi(x)$  and  $\exists x\varphi(x)$  are formulas, where  $\varphi(x)$  is obtained from  $\varphi(a)$  by substituting  $x$  for all  $a$  in  $\varphi(a)$ .

A formula without any occurrence of free variables is called a **sentence**. A formula which appears in the construction of a formula is called a **subformula**:

- (1) A formula  $\varphi$  is a subformula of  $\varphi$  itself.
- (2) formulas  $\varphi$  and  $\psi$  are subformulas of  $(\varphi \wedge \psi)$  and  $(\varphi \vee \psi)$ .
- (3)  $\varphi(a)$  is a subformula of  $\forall x\varphi(x)$  and  $\exists x\varphi(x)$ .
- (4) If  $\varphi$  is a subformula of  $\psi$ , then a subformula of  $\varphi$  is a subformula of  $\psi$ .

Formulas are denoted by  $\varphi, \psi, \dots; \varphi(a), \psi(a), \dots$ .

## Sequents

A formal expressions of the form

$$\varphi_1, \dots, \varphi_m \Rightarrow \psi_1, \dots, \psi_n,$$

where  $\varphi_1, \dots, \varphi_m; \psi_1, \dots, \psi_n$  are formulas, is called a **sequent**. The sequence  $\varphi_1, \dots, \varphi_m$  is called the **antecedent**, and the sequence  $\psi_1, \dots, \psi_n$  the **succedent** of the sequent.

Finite sequences of formulas are denoted by  $\Gamma, \Delta, \dots$ . So sequents are written as the form  $\Gamma \Rightarrow \Delta$ .

An inference is an expression of the form

$$\frac{S_1}{S} \quad \text{or} \quad \frac{S_1 \quad S_2}{S},$$

where  $S_1, S_2$  and  $S$  are sequents.  $S_1$  and  $S_2$  are called the **upper sequents** and  $S$  is called the **lower sequent** of the inference.

#### 4.1.2 Inference rules of LK and LJ

The difference between the classical logic LK and intuitionistic logic LJ, is secured by the intuitionistic restriction stated for two of the postulates: introduction of thinning and negation  $\neg$ .

A **proof** is constructed according to the following rules, which regulate the logical symbols  $\supset, \wedge, \vee, \forall$  and  $\exists$ .

**Beginning sequents:** **Logical axiom** is a sequent of the form  $\varphi \Rightarrow \varphi$ . Every proof in LK and LJ starts with logical axiom(s).

**Structural rules :**

$$\begin{array}{l} \text{Thinning :} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \\ \text{where } \Delta \text{ is empty for} \\ \text{the intuitionistic logic} \end{array}$$

$$\text{Contraction :} \quad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$\text{Interchange :} \quad \frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda}$$

$$\text{Cut :} \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

**Logical rules:**

$$\supset : \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Lambda}{\varphi \supset \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$\wedge : \frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$\vee : \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\neg : \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg \varphi, \Gamma \Rightarrow \Delta}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \varphi}$$

where  $\Delta$  is empty for  
the intuitionistic logic

$$\forall : \frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta}$$

where  $t$  is any term

$$\frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)}$$

where  $a$  is a free variable which does  
not occur in the lower sequent.

$$\exists : \frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x \varphi(x), \Gamma \Rightarrow \Delta}$$

where  $a$  is a free variable which does  
not occur in the lower sequent.

$$\frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)}$$

where  $t$  is any term

**DEFINITION 4.1.1.** We denote “a sequent  $\Gamma \Rightarrow \Delta$  is provable in LK ” by

$$\text{LK} \vdash \Gamma \Rightarrow \Delta,$$

and denote “a sequent  $\Gamma \Rightarrow \Delta$  is provable in LJ ” by

$$\text{LJ} \vdash \Gamma \Rightarrow \Delta,$$

Note that if  $\text{LJ} \vdash \Gamma \Rightarrow \Delta$  then  $\text{LK} \vdash \Gamma \Rightarrow \Delta$ .

$\text{LJ} \vdash \varphi \Leftrightarrow \psi$  is an abbreviation of “(  $\text{LJ} \vdash \varphi \Rightarrow \psi$ ) and (  $\text{LJ} \vdash \psi \Rightarrow \varphi$  )”. For example, we have following theorems.

**THEOREM 4.1.1.** For arbitrary formulas  $\varphi, \psi, \theta$ ,

- (1)  $\text{LJ} \vdash \varphi \wedge (\psi \vee \xi) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \xi)$
- (2)  $\text{LJ} \vdash \varphi \vee (\psi \wedge \xi) \Leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \xi)$
- (3)  $\text{LJ} \vdash \varphi \wedge \exists x\psi(x) \Leftrightarrow \exists x(\varphi \wedge \psi(x))$
- (4)  $\text{LJ} \vdash \varphi \vee \forall x\psi(x) \Leftrightarrow \forall x(\varphi \vee \psi(x))$

*Proof.* (1)

$$\frac{\frac{\varphi, \psi \Rightarrow \varphi \wedge \psi}{\varphi, \psi \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \xi)} \quad \frac{\varphi, \xi \Rightarrow \varphi \wedge \xi}{\varphi, \xi \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \xi)}}{\varphi, (\psi \vee \xi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \xi)} \text{ (}\vee\text{-left)}$$

$$\frac{\frac{\vdots}{\varphi \wedge \psi \Rightarrow \varphi \wedge (\psi \vee \xi)} \quad \frac{\vdots}{\varphi \wedge \xi \Rightarrow \varphi \wedge (\psi \vee \xi)}}{(\varphi \wedge \psi) \vee (\varphi \wedge \xi) \Rightarrow \varphi \wedge (\psi \vee \xi)}$$

(2) Similar to (1).

(3)

$$\frac{\frac{\frac{\vdots}{\varphi, \psi(a) \Rightarrow \varphi \wedge \psi(a)}}{\varphi, \psi(a) \Rightarrow \exists x(\varphi \wedge \psi(x))}}{\frac{\varphi, \exists x\psi(x) \Rightarrow \exists x(\varphi \wedge \psi(x))}{\varphi \wedge \exists x\psi(x) \Rightarrow \exists x(\varphi \wedge \psi(x))}}$$

$$\frac{\frac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi(a) \Rightarrow \varphi} \quad \frac{\vdots}{\varphi \wedge \psi(a) \Rightarrow \exists x\psi(x)}}{\frac{\varphi \wedge \psi(a) \Rightarrow \varphi \wedge \exists x\psi(x)}{\exists x(\varphi \wedge \psi(x)) \Rightarrow \varphi \wedge \exists x\psi(x)}}$$

(4)

$$\begin{array}{c}
 \vdots \\
 \hline
 \varphi \vee \psi(a) \Rightarrow \varphi, \psi(a) \\
 \hline
 \forall x(\varphi \vee \psi(x)) \Rightarrow \varphi, \psi(a) \\
 \hline
 \forall x(\varphi \vee \psi(x)) \Rightarrow \varphi, \forall x\psi(x) \\
 \hline
 \forall x(\varphi \vee \psi(x)) \Rightarrow \varphi \vee \forall x\psi(x) \\
 \\
 \begin{array}{cc}
 \varphi \Rightarrow \varphi & \vdots \\
 \hline
 \varphi \Rightarrow \varphi \vee \psi(a) & \forall x\psi(x) \Rightarrow \varphi \vee \psi(a) \\
 \hline
 \varphi \vee \forall x\psi(x) \Rightarrow \varphi \vee \psi(a) \\
 \hline
 \varphi \vee \forall x\psi(x) \Rightarrow \forall x(\varphi \vee \psi(x))
 \end{array}
 \end{array}$$

□

Also we have:

**THEOREM 4.1.2.** *For arbitrary formulas  $\varphi, \psi, \theta$ ,*

- (1)  $\text{LJ} \vdash (\varphi \wedge \neg\varphi) \Rightarrow \theta$ ,
- (2)  $\text{LK} \vdash \theta \Rightarrow (\varphi \vee \neg\varphi)$ ,
- (3)  $\text{LK} \vdash \varphi \Leftrightarrow \neg\neg\varphi$ ,
- (4)  $\text{LK} \vdash \neg(\varphi \vee \psi) \Leftrightarrow (\neg\varphi \wedge \neg\psi)$ ,
- (5)  $\text{LK} \vdash (\neg\varphi \vee \neg\psi) \Leftrightarrow \neg(\varphi \wedge \psi)$ .
- (6)  $\text{LK} \vdash (\varphi \supset \psi) \Leftrightarrow \neg\varphi \vee \psi$

G.Gentzen proved in [3], the following ‘Hauptsatz’ (Cut Elimination Theorem) for LK, and it follows that LK is consistent.

**THEOREM 4.1.3** (Hauptsatz). *If a sequent  $\Gamma \Rightarrow \Delta$  is provable in LK, then  $\Gamma \Rightarrow \Delta$  is provable without using **Cut**, in LK.*

*Hauptsatz also holds for LJ.*

**COROLLARY 4.1.4** (Consistency). *Formulas of the form  $\varphi \wedge \neg\varphi$ , which represents a contradiction, is not provable in LK.*

### 4.1.3 Axioms of set theory

Set theory ZFC means the logical system LK with predicate constants  $\in$  and  $=$ , and the following axioms A1-A9, AC by Zermelo and Freankel, where  $\varphi \equiv \psi$  is an abbreviation of  $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$ .

**A1. Equality**  $\forall u \forall v ((u = v \wedge \varphi(u)) \supset \varphi(v))$ .

**A2. Extensionality**  $\forall u, v (\forall x (x \in u \equiv x \in v) \supset (u = v))$ .

**A3. Pairing**  $\forall u, v \exists z (\forall x (x \in z \equiv (x = u \vee x = v)))$ .

The set  $z$  satisfying  $\forall x (x \in z \equiv (x = u \vee x = v))$  is denoted by  $\{u, v\}$ .

**A4. Union**  $\forall u \exists z (\forall x (x \in z \equiv \exists y \in u (x \in y)))$ .

The set  $z$  satisfying  $\forall x (x \in z \equiv \exists y \in u (x \in y))$  is denoted by  $\bigcup u$ .

**A5. Power set**  $\forall u \exists z (\forall x ((x \in z) \equiv (x \subset u)))$ , where

$$x \subset u \stackrel{\text{def}}{\iff} \forall y ((y \in x) \supset (y \in u)).$$

The set  $z$  satisfying  $\forall x ((x \in z) \equiv (x \subset u))$  is denoted by  $\mathcal{P}(u)$ .

**A6. Infinity**  $\exists u (\exists x (x \in u) \wedge \forall x \in u \exists y \in u (x \in y))$ .

**A7. Separation**  $\forall u \exists v \forall x ((x \in v) \equiv (x \in u \wedge \varphi(x)))$ .

The set  $v$  satisfying  $\forall x ((x \in v) \equiv (x \in u \wedge \varphi(x)))$  is denoted by  $\{x \in u \mid \varphi(x)\}$ .

**A8. Collection**  $\forall u \exists v ((\forall x \in u \exists y \varphi(x, y)) \supset \forall x \in u \exists y \in v \varphi(x, y))$ .

**A9.  $\in$ -induction**  $\forall x (\forall y \in x \varphi(y)) \supset \varphi(x)$ .

**AC (Axiom of choice)** If  $u$  is a set of nonempty sets, there exists a function  $f$  such that for every  $x \in u$ ,  $f(x) \in x$ .

$$\forall u \left( (u \neq \emptyset \wedge \forall x \in u (x \neq \emptyset)) \supset \exists f \subset (u \times \bigcup u) (\forall x \in u (f(x) \in x)) \right)$$

The Axiom of choice is known to be equivalent to the following Axiom of Zorn, under the classical logic.

**Zorn (Zorn's lemma)**  $\forall v(\text{Chain}(v, u) \supset (\bigcup v \in u)) \supset \exists z \text{Max}(z, u)$ , where

$$\begin{aligned}\text{Chain}(v, u) &\stackrel{\text{def}}{\iff} (v \subset u) \wedge \forall x, y \in v((x \subset y) \vee (y \subset x)), \\ \text{Max}(z, u) &\stackrel{\text{def}}{\iff} (z \in u) \wedge \forall x((x \in u) \wedge (z \subset x)) \supset (z = x).\end{aligned}$$

The entire sets form a universe of set theory. Since axioms  $A_1, \dots, A_9, AC$  are true in the universe  $V$ (p.78), every formula induced from these axioms by LK is also true. Note that  $\forall x((x \in u) \supset \varphi(x))$  and  $\exists x((x \in u) \wedge \varphi(x))$  are shorten as  $\forall x \in u \varphi(x)$  and  $\exists x \in u \varphi(x)$ , respectively.

LK with the system of axioms  $\{A_1, \dots, A_9, AC\}$  is called the formal system of **classical set theory** and denoted by ZFC; LJ with the axiom system  $\{A_1, \dots, A_9, \text{Zorn}\}$  is called the formal system of **intuitionistic set theory** and denoted by IZFZ.

A sequent  $\Gamma \Rightarrow \Delta$  is said to be **provable in ZFC**, in symbols

$$\text{ZFC} \vdash \Gamma \Rightarrow \Delta$$

if

$$\text{LK} \vdash (A_1, \dots, A_9, AC, \Gamma) \Rightarrow \Delta.$$

$\text{ZFC} \vdash \Rightarrow \varphi$  is shorten as  $\text{ZFC} \vdash \varphi$ . If ZFC is trivial in the context, shorten as  $\vdash \varphi$ .

A sequent  $\Gamma \Rightarrow \Delta$  is said to be **provable in IZFZ**, in symbols

$$\text{IZFZ} \vdash \Gamma \Rightarrow \Delta$$

if

$$\text{LJ} \vdash (A_1, \dots, A_9, \text{Zorn}, \Gamma) \Rightarrow \Delta.$$

$\text{IZFZ} \vdash \Rightarrow \varphi$  is shorten as  $\text{IZFZ} \vdash \varphi$ .

ZFC is said to be **inconsistent** if the empty sequent  $\Rightarrow$  is provable in ZFC, otherwise **consistent**.

## 4.2 Construction of mathematics in ZFC

### 4.2.1 Definition of sets

If  $\varphi(x)$  is a predicate with 1-argument and

$$\text{ZFC} \vdash \exists! x \varphi(x) \wedge \varphi(u), \quad \text{where}$$

$$\exists! x \varphi(x) \stackrel{\text{def}}{\iff} \exists x \varphi(x) \wedge \forall x, y (\varphi(x) \wedge \varphi(y) \supset (x = y)),$$

then the set of  $x$  satisfying  $\varphi(x)$  is defined and denoted by  $\{x \mid \varphi(x)\}$ . For example, pair  $\{u, v\}$  is a set defined by Axiom A3 (Pairing).

$$\{u, v\} = \{x \mid x = u \vee x = v\}$$

### 4.2.2 Ordered pairs

Let

$$\langle x, y \rangle \stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\}, \quad \text{where } \{x\} \text{ is an abbreviation of } \{x, x\}$$

.

**LEMMA 4.2.1.**  $\langle x, y \rangle$  satisfies

$$\forall u, v \left( (\langle x, y \rangle = \langle u, v \rangle) \equiv ((x = u) \wedge (y = v)) \right) \quad (4.2.1)$$

*Proof.*  $\vdash ((x = u) \wedge (y = v)) \Rightarrow (\langle x, y \rangle = \langle u, v \rangle)$  is obvious. The converse is proved as follows.

$$\begin{aligned} \vdash \langle x, y \rangle = \langle u, v \rangle &\Rightarrow \{x\} \in \langle u, v \rangle \\ &\Rightarrow (\{x\} = \{u\}) \vee (\{x\} = \{u, v\}) \\ &\Rightarrow (x = u) \vee (x = u = v) \\ &\Rightarrow (x = u) \\ \therefore \vdash \langle x, y \rangle = \langle u, v \rangle &\Rightarrow (x = u) \wedge (\{x, y\} \in \langle u, v \rangle) \\ &\Rightarrow (x = u) \wedge ((y \in \{u\}) \vee (y \in \{u, v\})) \\ &\Rightarrow ((x = u) \wedge (y = u)) \vee ((x = u) \wedge (y \in \{u, v\})) \\ &\Rightarrow (x = u = y) \vee ((x = u) \wedge ((y = u) \vee (y = v))) \\ &\Rightarrow (x = u = y) \vee ((x = u) \wedge (y = v)) \\ &\Rightarrow (x = u = y = v) \vee ((x = u) \wedge (y = v)) \\ &\Rightarrow (x = u) \wedge (y = v) \end{aligned}$$

□

$\langle x, y \rangle$  is called an **ordered pair** and also denoted by  $(x, y)$ .

### 4.2.3 Relations

The set of all ordered pairs of elements of  $X$  and  $Y$  is denoted by  $X \times Y$ .

$$X \times Y \stackrel{\text{def}}{=} \{\langle x, y \rangle \mid x \in X, y \in Y\}$$

A subset of  $X \times Y$  is called a **relation**. Especially a subset of  $X \times X$  is said to be a relation on  $X$ .

If  $F \subset X \times Y$ , then  $\langle x, y \rangle \in F$  is denoted by  $F(x, y)$  or  $xFy$ .

#### Order relation

A relation  $\leq$  on an set  $X$  is said to be an **order relation on  $X$** , if

- (1)  $\forall x \in X (x \leq x)$  (reflexive)
- (2)  $\forall x, y \in X ((x \leq y \wedge y \leq x) \supset (x = y))$  (antisymmetric)
- (3)  $\forall x, y, z \in X ((x \leq y \wedge y \leq z) \supset (x \leq z))$  (transitive)

### 4.2.4 Functions

A relation  $f \subset X \times Y$  is called a **function** from  $X$  to  $Y$ , in symbols  $f : X \rightarrow Y$ , if for each  $x \in X$  there exists a unique  $y$  such that  $\langle x, y \rangle \in f$ .

$$(f : X \rightarrow Y) \stackrel{\text{def}}{\iff} (f \subset X \times Y) \wedge \forall x \in X \exists! y (\langle x, y \rangle \in f), \quad \text{where}$$

$$\exists! y (\langle x, y \rangle \in f) \stackrel{\text{def}}{\iff} \exists y (\langle x, y \rangle \in f) \wedge \forall y, z \in Y ((\langle x, y \rangle \in f \wedge \langle x, z \rangle \in f) \supset y = z).$$

If  $f$  is a function, then  $\langle x, y \rangle \in f$  is also denoted by

$$f(x) = y \quad \text{or} \quad f : x \mapsto y$$

### 4.2.5 Equivalence relation

**DEFINITION 4.2.1.** A relation  $\equiv$  on a set  $X$  is called an **equivalence relation** if the following conditions are satisfied for  $x, y, z \in X$ .

- (1)  $\vdash x \equiv x$ ,
- (2)  $\vdash x \equiv y \Rightarrow y \equiv x$ ,

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$$(3) \quad \vdash x \equiv y \wedge y \equiv z \Rightarrow x \equiv z.$$

The set of elements equivalent to  $a \in X$  is called an **equivalence class** and denoted by  $|a|$ .

$$|a| \stackrel{\text{def}}{=} \{x \in X \mid x \equiv a\}$$

The set of all equivalence classes is called the **quotient** and denoted by  $X/\equiv$ .

$$X/\equiv \stackrel{\text{def}}{=} \{|x| \subset X \mid x \in X\}$$

If A relation  $R(x_1, \dots, x_n)$  on  $X$  satisfies

$$\vdash R(x_1, \dots, x_n) \wedge \forall i (x_i \equiv x'_i) \Rightarrow R(x'_1, \dots, x'_n)$$

then the relation  $R(x_1, \dots, x_n)$  on  $X$  induces a relation on  $X/\equiv$ :

$$\vdash R(x_1, \dots, x_n) \Leftrightarrow R(|x_1|, \dots, |x_n|).$$

Similarly, if

$$\vdash f \text{ is a function and}$$

$$\vdash \forall i (x_i \equiv x'_i) \Rightarrow f(x_1, \dots, x_n) \equiv f(x'_1, \dots, x'_n),$$

then a function  $f$  on  $X$  induces a function on  $X/\equiv$ .

#### 4.2.6 Natural numbers

**LEMMA 4.2.2.** (1) *There exists a unique empty set denoted by  $\emptyset$ .*

$$\vdash \forall x (x \notin \emptyset) \wedge \forall y (\forall x (x \notin y) \supset (y = \emptyset)).$$

*Proof. Existence :* By Axiom 6 (Infinity), there exists at least one set. Say  $u$ . By Axiom 7 (Separation), there exists set  $\{x \in u \mid x \neq x\}$ , which has no element. Let

$$\emptyset \stackrel{\text{def}}{=} \{x \in u \mid x \neq x\}.$$

**Uniqueness :**  $\forall v (\forall x (x \notin v) \supset (v = \emptyset))$  is obvious. □

(2) *For a set  $u$ , the singleton  $\{u\}$  consisting of only  $u$  is defined by*

$$\{u\} \stackrel{\text{def}}{=} \{u, u\}.$$

- (3) The set  $\{u_1, \dots, u_n\}$  consisting of  $u_1, \dots, u_n$  is defined by using Axiom 4 (Union).

$$\{u_1, \dots, u_n\} \stackrel{\text{def}}{=} \{u_1\} \cup \dots \cup \{u_n\}.$$

**DEFINITION 4.2.2.**

$$\begin{aligned} 0 &\stackrel{\text{def}}{=} \emptyset \\ 1 &\stackrel{\text{def}}{=} \{0\} \\ 2 &\stackrel{\text{def}}{=} 1 \cup \{1\} = \{0, 1\} \\ &\vdots \\ S(n) &\stackrel{\text{def}}{=} n \cup \{n\} \end{aligned}$$

$0, 1, 2, \dots, n, \dots$  are called **natural numbers**.

**THEOREM 4.2.3.** The set of all natural numbers,  $\mathbb{N}$ , is defined so that

$$\vdash \forall n \left( (n \in \mathbb{N}) \equiv (n \mid n = 0 \vee \exists m \in n (n = S(m))) \right).$$

*Proof.* By using A6 (Infinity), there exists  $u$  and  $x_0$  such that

$$(x_0 \in u) \wedge \forall x (x \in u \supset \exists y \in u (x \in y)).$$

By Axiom 10 (Axiom of choice), there exists a function  $f : u \rightarrow \bigcup u$  such that

$$\forall x \in u (f(x) \in x).$$

Hence, there exist a sequence  $f(x_0), f(f(x_0)), f(f(f(x_0))), \dots$  in  $u$ . Let

$$\begin{aligned} f^1(x_0) &\stackrel{\text{def}}{=} f(x_0) \in u \\ f^2(x_0) &\stackrel{\text{def}}{=} f(f(x_0)) \in u \\ &\vdots \\ f^n(x_0) &\stackrel{\text{def}}{=} f(f^{n-1}(x_0)) \in u \\ &\vdots \\ \Sigma &\stackrel{\text{def}}{=} \{x_0, f^1(x_0), f^2(x_0), \dots, f^n(x_0), \dots\}, \end{aligned}$$

where  $\Sigma$  is constructed in ZFC as follows.

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By A5 (Power set), there exists the power set  $\mathcal{P}(u)$  of  $u$ , and

$$\{x_0, f^1(x_0), f^2(x_0), \dots, f^n(x_0)\} \in \mathcal{P}(u).$$

Using A7 (Separation), let

$$S(X) \stackrel{\text{def}}{=} \{x_0\} \cup \{f(t) \mid t \in X\},$$

$$\Sigma \stackrel{\text{def}}{=} \{x \in \mathcal{P}(u) \mid (\exists y \in x)(x = S(y))\}.$$

Now let

$$\varphi(x, y) \stackrel{\text{def}}{=} (x = \{x_0\} \wedge y = 0) \vee (x \in \Sigma \wedge y = S(\varphi(x))).$$

By A8 (Collection), there exists  $v$  such that

$$\forall t \in \Sigma \exists x \in v (x = \varphi(t)).$$

Hence, there exists  $\{x \in v \mid \exists t \in \Sigma (x = \varphi(t))\}$ , and

$$\vdash \left( (t \in \Sigma) \wedge (x = \varphi(t)) \right) \Leftrightarrow \left( (x \in v) \wedge (x = 0 \vee (\exists y \in x)(x = S(y))) \right).$$

Let

$$\mathbb{N} \stackrel{\text{def}}{=} \{x \in v \mid \exists t \in \Sigma (x = \varphi(t))\}.$$

Then

$$\vdash n \in \mathbb{N} \Leftrightarrow ((n = 0) \vee (\exists m \in n)(n = S(m))).$$

Also,

$$\vdash \forall x \in z \left( (0 \in z) \vee \exists y \in z (x = S(y)) \right) \Rightarrow (z = \mathbb{N}).$$

□

**DEFINITION 4.2.3.** *The following  $P1, \dots, P5$  are called **Peano's axioms**.*

**P1**  $\Rightarrow 0 \in \mathbb{N}$ .

**P2**  $x \in \mathbb{N} \Rightarrow S(x) \in \mathbb{N}$ .

**P3**  $S(x) = S(y) \Rightarrow x = y$ .

**P4**  $x \in \mathbb{N} \Rightarrow \neg(S(x) = 0)$ .

**P5**  $\left(0 \in M \wedge \forall x \in M (S(x) \in M)\right) \Rightarrow \mathbb{N} \subseteq M.$

**Peano arithmetics**, denoted by  $\text{PA}$ , is the theory based on  $\text{LK}$  with Peano's axioms.

A theory  $\mathfrak{S}$  is called an **extension of**  $\text{PA}$  if every theorem of  $\text{PA}$  is provable in  $\mathfrak{S}$ .

**THEOREM 4.2.4.**  $\text{ZFC}$  is an extension of  $\text{PA}$ , i.e.

$$\text{ZFC} \vdash P1 \wedge \dots \wedge P5$$

*Proof.* (1)  $\text{ZFC} \vdash P1$  :

$$\frac{\begin{array}{c} \Rightarrow a \in 0 \equiv a \in 0 \\ \hline \Rightarrow \forall x (x \in 0 \equiv x \in 0) \quad \text{i.e.} \quad \Rightarrow 0 = 0 \\ \hline \Rightarrow 0 = 0 \vee \exists m \in 0 (0 = S(m)) \\ \hline \Rightarrow 0 \in \mathbb{N} \end{array}}$$

Therefore,  $\text{ZFC} \vdash 0 \in \mathbb{N}$ .

(2)  $\text{ZFC} \vdash P2$  :

If  $n \in \mathbb{N}$ , then  $n \in (n \cup \{n\}) = S(n)$ .

$$\therefore \exists m \in S(n) (S(m) = S(n)). \quad \therefore S(n) \in \mathbb{N}.$$

**LEMMA 4.2.5.**  $\text{ZFC} \vdash (n \in \mathbb{N}) \wedge (m \in \mathbb{N}) \wedge (m \in n) \Rightarrow (m \subset n)$ .

*Proof.* Let  $\varphi(n)$  be  $(n \in \mathbb{N}) \supset \forall m \in \mathbb{N} (m \in n \supset m \subset n)$  and use A9( $\in$ -induction).

$$\begin{aligned} (n \in \mathbb{N}) \wedge (m \in \mathbb{N}) &\wedge \forall m \in n \varphi(m) \wedge (m \in n) \wedge (t \in m) \\ &\Rightarrow \exists x \in n \left( \varphi(x) \wedge (n = S(x)) \wedge (m \in n) \wedge (t \in m) \right) \\ &\Rightarrow \exists x \left( \varphi(x) \wedge (x \in n) \wedge (m \in x \vee m = x) \wedge (t \in m) \right) \\ &\Rightarrow \exists x \left( (x \subset n) \wedge (m \in x \vee m = x) \wedge (t \in m) \right) \\ &\Rightarrow t \in n \end{aligned}$$

$$\therefore (n \in \mathbb{N}) \wedge (m \in \mathbb{N}) \wedge \forall t \in n \varphi(t) \wedge m \in n \Rightarrow m \subset n$$

$$\therefore (n \in \mathbb{N}) \wedge \forall m \in n \varphi(m) \Rightarrow \varphi(n).$$

By A9,  $\forall n \varphi(n)$ , i.e.

$$\text{ZFC} \vdash (n, m \in \mathbb{N}) \wedge (m \in n) \Rightarrow m \subset n \quad (4.2.2)$$

□

(3)  $\text{ZFC} \vdash P3 :$

*Proof.* If  $S(m) = S(n)$  and  $t \in n$ , since  $n \in S(n) = S(m)$ , then  $n \in m$  or  $n = m$ .

If  $n \in m$ , then  $n \subset m$  by (4.2.2). Therefore,

$$(S(n) = S(m)) \wedge (t \in n) \Rightarrow (t \in m).$$

Similarly,

$$(S(n) = S(m)) \wedge (t \in m) \Rightarrow (t \in n).$$

□

(4)  $\text{ZFC} \vdash P4 :$

*Proof.*

$$(n = 0) \vee \exists m (n = S(m)). \quad \therefore \exists x (x \in S(n)).$$

$$\therefore (n \in \mathbb{N}) \wedge (S(n) = 0) \Rightarrow \perp.$$

$$\therefore (n \in \mathbb{N}) \Rightarrow \neg(S(n) = 0).$$

□

(5)  $\text{ZFC} \vdash P5 :$

*Proof.* Let  $\varphi(n)$  be  $(n \in \mathbb{N}) \supset (n \in M)$ .

$$(n \in \mathbb{N}) \wedge \forall m \in n \varphi(m) \Rightarrow \exists x \in M (n = S(x))$$

$$(n \in \mathbb{N}) \wedge \forall m \in n \varphi(m) \Rightarrow n \in M$$

$$\therefore \forall m \in n \varphi(m) \Rightarrow \varphi(n)$$

$$\therefore \text{ZFC} \vdash \left( (0 \in M) \wedge \forall x ((x \in M) \supset (S(x) \in M)) \right) \supset (\mathbb{N} \subset M).$$

By A9,  $\text{ZFC} \vdash \forall n \varphi(n)$ .

□

□

**COROLLARY 4.2.6.** *The following inference is valid.*

$$[P5'] \quad \frac{\varphi(a), \Gamma \Rightarrow \Delta, \varphi(S(a))}{\varphi(0), \Gamma \Rightarrow \Delta, \forall x \in \mathbb{N} \varphi(x)}.$$

## 4.2.7 Operations on the natural numbers

Sum  $+$  and product  $\cdot$  are defined as functions on  $\mathbb{N} \times \mathbb{N}$  to  $\mathbb{N}$ , where  $+(\langle x, y \rangle)$  is denoted by  $x + y$ , and  $\cdot(\langle x, y \rangle)$  is denoted by  $x \cdot y$ .

$+$  is defined by

$$\begin{cases} 0 + b = b \\ S(a) + b = S(a + b) \end{cases} \quad (4.2.3)$$

$\cdot$  is defined by

$$\begin{cases} 0 \cdot b = 0 \\ S(a) \cdot b = a \cdot b + b \end{cases} \quad (4.2.4)$$

The definability of  $+$  and  $\cdot$  on  $\mathbb{N}$  are provable in ZFC, using Peano's axioms.

The following theorems are provable in ZFC.

**THEOREM 4.2.7** (Properties of  $+$ ).

- (1)  $(a + b) + c = a + (b + c)$  [associative]
- (2)  $S(a + b) = a + S(b)$
- (3)  $a + b = b + a$  [commutative]
- (4)  $S(a) = a + 1$
- (5)  $a + b = a + c \Rightarrow b = c$
- (6)  $a + b = a \Rightarrow b = 0$
- (7)  $a + b = 0 \Rightarrow a = 0 \wedge b = 0$
- (8)  $a + b = 1 \Rightarrow a = 1 \vee b = 1$

$$(9) \quad a \neq b \Rightarrow \exists x(a = b + x) \vee \exists y(b = a + y)$$

*Proof.* (1) Let  $P(x)$  be the formula  $(x + b) + c = x + (b + c)$ , and prove, by induction,

$$\text{ZFC} \vdash \quad \Rightarrow \forall x((x + b) + c = x + (b + c)).$$

$$(a) \quad \text{ZFC} \vdash \quad \Rightarrow P(0), \text{ because, } (0 + b) + c = b + c = 0 + (b + c).$$

$$(b) \quad \text{ZFC} \vdash P(x) \Rightarrow P(S(x)).$$

$$\begin{aligned} \therefore) \quad (S(x) + b) + c &= S(x + b) + c \quad [\text{definition of } +] \\ &= S((x + b) + c) \quad [\text{definition of } +] \\ &= S(x + (b + c)) \quad [\text{hypothesis of induction}] \\ &= S(x) + (b + c) \quad [\text{definition of } +] \end{aligned}$$

$$\therefore \vdash P(x) \Rightarrow P(S(x))$$

$$\therefore \vdash P(0) \wedge \forall x(P(x) \supset P(S(x)))$$

By P5(Induction),  $\forall x P(x)$ , i.e.

$$\vdash \forall x((x + b) + c = x + (b + c))$$

(2) Let  $P(x)$  be the formula  $x + S(b) = S(x + b)$ , and prove  $\forall x P(x)$  by induction.

$$(a) \quad \vdash P(0) \text{ is obvious.}$$

$$(b) \quad \vdash P(x) \Rightarrow P(S(x)) \text{ is proved as follows.}$$

$$\begin{aligned} S(x) + S(b) &= S(x + S(b)) \quad [\text{definition of } +] \\ &= S(S(x + b)) \quad [\text{hypothesis of induction}] \\ &= S(S(x) + b) \quad [\text{definition of } +] \end{aligned}$$

(3) Let  $P(x)$  be  $x + b = b + x$ , and use induction.

(a)  $P(0)$  is  $0 + b = b + 0$ . Since  $0 + b = b$  is provable by the definition of  $+$ , it suffices to show that  $b + 0 = b$ . Now we prove that  $\forall x \in \mathbb{N}(x + 0 = x)$  by induction.

Let  $P_1$  be the formula  $x + 0 = x$ .

- i.  $P_1(0)$  is provable by the definition of  $+$ .
- ii. Assume  $P_1(x)$ .

$$\begin{aligned} S(x) + 0 &= S(x + 0) && [\text{definition of } +] \\ &= S(x) && [\text{hypothesis } P_1(x) \text{ of induction}] \end{aligned}$$

- (b) Assume that  $P(x)$  and prove  $P(S(x))$ .

$$\begin{aligned} S(x) + b &= S(x + b) && [\text{definition of } +] \\ &= S(b + x) && [\text{hypothesis of induction}] \\ &= b + S(x) && [\text{by (2)}] \end{aligned}$$

- (4)  $\forall x(S(x) = x + 1)$  is provable, since

$$S(a) = S(a + 0) = a + S(0) = a + 1 \quad \text{by (2).}$$

- (5) Let  $P(x)$  be  $(x + b = x + c) \supset (b = c)$  and prove  $\forall x P(x)$  by induction.

- (a)  $P(0)$  is obviously provable.
- (b) Proof of  $P(x) \Rightarrow P(S(x))$  is as follows.

$$\begin{aligned} S(x) + b = S(x) + c &\Rightarrow S(x + b) = S(x + c) && [\text{definition of } +] \\ &\Rightarrow x + b = x + c && [\text{Peano's axiom P3}] \\ &\Rightarrow b = c && [\text{hypothesis of induction}] \end{aligned}$$

- (6) If  $x \in \mathbb{N}$  and  $x \neq 0$ , then  $x = S(y)$  for some  $y \in \mathbb{N}$ . Let  $P(x)$  be  $\neg(x + S(a) = x)$ , and prove  $\forall x \in \mathbb{N} \neg P(x)$  by induction.

- (a)  $P(0) : \neg(0 + S(a) = 0)$  is provable by Peano's axiom  $P(4)$ .
- (b)  $P(x) \Rightarrow P(S(x))$  is provable as follows.

$$\begin{aligned} S(x) + S(a) = S(x) &\Rightarrow S(x + S(a)) = S(x) \\ &\Rightarrow x + S(a) = x && [\text{by Peano's axiom P3}] \\ \therefore \neg(x + S(a) = x) &\Rightarrow \neg(S(x) + S(a) = S(x)) \\ \therefore P(x) &\Rightarrow P(S(x)) \end{aligned}$$

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(7) We prove the contraposition of  $(a + b = 0) \Rightarrow (a = 0 \wedge b = 0)$ , i.e.

$$\neg(a = 0 \wedge b = 0) \Rightarrow \neg(a + b = 0).$$

$$\begin{aligned} \neg(a = 0 \wedge b = 0) &\Rightarrow \neg(a = 0) \vee \neg(b = 0) \\ &\Rightarrow \exists c, d (a = S(c) \vee b = S(d)) \\ a = S(c) &\Rightarrow a + b = S(c) + b \\ &\Rightarrow a + b = S(c + b) \\ &\Rightarrow \neg(a + b = 0) \quad [\text{Peano's axiom P4}] \\ b = S(d) &\Rightarrow a + b = b + a \quad [\text{commutative law of +}] \\ &\Rightarrow a + b = S(d) + a = S(d + a) \\ &\Rightarrow \neg(a + b = 0) \\ \therefore \neg(a = 0 \wedge b = 0) &\Rightarrow \neg(a + b = 0) \end{aligned}$$

(8) If  $a = 0$ , then  $b = 0 + b = a + b = 1$ . Hence

$$(a + b = 1) \wedge (a = 0) \Rightarrow (b = 1)$$

If  $\neg(a = 0)$ , then  $a = S(c)$  for some  $c \in \mathbb{N}$ .

$$\begin{aligned} a + b = 1 \wedge a = S(c) &\Rightarrow S(c) + b = 1 \\ &\Rightarrow S(c + b) = S(0) \\ &\Rightarrow c + b = 0 \quad [\text{Peano's axiom P4}] \\ &\Rightarrow c = 0 \wedge b = 0 \quad [(7)] \\ &\Rightarrow a = S(c) = S(0) = 1 \\ \therefore (a + b = 1) \wedge \neg(a = 0) &\Rightarrow (a = 1) \\ \text{By } \vdash (a = 0) \vee \neg(a = 0) \text{ and distributive law,} \\ a + b = 1 &\Rightarrow a + b = 1 \wedge (a = 0 \vee \neg(a = 0)) \\ &\Rightarrow (a + b = 1 \wedge a = 0) \vee (a + b = 1 \wedge \neg(a = 0)) \\ &\Rightarrow a = 1 \vee b = 1 \end{aligned}$$

(9) (a) If  $b = 0$ ,  $a = b + a$ .  $\therefore \exists x (a = b + x)$ .

If  $b \neq 0$ , then  $b = S(c) = c + 1$  for some  $c \in \mathbb{N}$ . Since  $a \neq b = c + 1$ ,  
 $a = c$  or  $a \neq c$ .

- i. If  $a = c$ , then  $a + 1 = c + 1 = b$ , hence  $\exists y(b = a + y)$ .
- ii. If  $a \neq c$ , By the hypothesis of induction

$$\vdash a \neq c \Rightarrow \exists x(a = c + x) \vee \exists y(c = a + y).$$

$$\therefore \exists x(a = c + x) \vee \exists y(c = a + y)$$

Let  $\exists x(a = c + x)$ .

If  $x = 0$ , then  $a = c \quad \therefore a + 1 = c + 1 = b$

If  $x \neq 0$ , then  $x = S(z) = z + 1$ , hence

$$a = c + x = c + (z + 1) = (c + 1) + z = b + z$$

$$\therefore \exists x(a = b + x) \vee \exists y(b = a + y).$$

Similar for the case  $\exists y(c = a + y)$ .

□

The operation  $\cdot$  is defined by

$$\begin{cases} 0 \cdot b = 0 \\ S(a) \cdot b = a \cdot b + b. \end{cases} \quad (4.2.5)$$

**THEOREM 4.2.8** (Properties of  $\cdot$ ).

- (1)  $0 \cdot a = a \cdot 0 = 0$ .
- (2)  $1 \cdot a = a \cdot 1 = a$ .
- (3)  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ .
- (4)  $a \cdot b = b \cdot a$ .
- (5)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ .
- (6)  $a \cdot b = 0 \Rightarrow (a = 0 \vee b = 0)$ .
- (7)  $a \cdot b = 1 \Rightarrow (a = 1 \wedge b = 1)$ .
- (8)  $a \neq 0 \wedge (a \cdot b = a \cdot c) \Rightarrow b = c$ .

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$$(9) \quad a \neq 0 \wedge (a \cdot b = a) \Rightarrow b = 1.$$

*Proof.*

(1) Let  $P(x)$  be the formula  $x \cdot 0 = 0$ , and we prove  $\forall x P(x)$  by induction .

(a)  $P(0)$ , i.e.  $0 \cdot 0 = 0$  by definition.

(b)  $\vdash P(x) \Rightarrow P(S(x))$ , because

$$S(x) \cdot 0 = x \cdot 0 + 0 = 0 + 0 = 0.$$

$\therefore a \cdot 0 = 0$ .  $0 \cdot a = 0$  by the definition of  $\cdot$ .

(2)  $1 \cdot a = S(0) \cdot a = 0 \cdot a + a = a$ .  $\therefore 1 \cdot a = a$ .

$\forall x(x \cdot 1 = x)$  is proved by induction.

Let  $P(x)$  be the formula  $x \cdot 1 = x$ .

(a)  $P(0)$  is obvious from the definition.

(b)  $\vdash P(x) \Rightarrow P(S(x))$ , because

$$S(x) \cdot 1 = x \cdot 1 + 1 = x + 1 = S(x)$$

(3) Let  $P(X)$  be  $x \cdot (b+c) = (x \cdot b) + (x \cdot c)$ , and prove  $\forall x P(x)$  by induction.

(a)  $P(0)$ , i.e.  $0 \cdot (b+c) = (0 \cdot b) + (0 \cdot c) = 0$  is obvious.

(b)  $\vdash P(x) \Rightarrow P(S(x))$ , because

$$\begin{aligned} S(x) \cdot (b+c) &= (x \cdot (b+c)) + (b+c) \\ &= (x \cdot b + x \cdot c) + (b+c) \quad [\text{hypothesis of induction}] \\ &= (x \cdot b + b) + (x \cdot c + c) \quad [\text{property of } +] \\ &= (S(x) \cdot b) + (S(x) \cdot c) \end{aligned}$$

(4) Let  $P(x)$  be  $x \cdot b = b \cdot x$  and prove  $\forall x P(x)$  by induction.

(a)  $P(0)$ , i.e.  $0 \cdot b = b \cdot 0$  is obvious by (1).

(b)  $\vdash P(x) \Rightarrow P(S(x))$ , because

$$\begin{aligned} S(x) \cdot b &= x \cdot b + b \\ b \cdot S(x) &= b \cdot (x + 1) = b \cdot x + b \cdot 1 \quad \text{law of distribution} \\ &= x \cdot b + b \quad [\text{hypothesis } P(x)] \\ &= S(x) \cdot b \end{aligned}$$

$$\therefore b \cdot S(x) = S(x) \cdot b.$$

(5) Let  $P(x)$  be  $(x \cdot b) \cdot c = x \cdot (b \cdot c)$ , and prove  $\forall x P(x)$  by induction.

(a)  $P(0)$ , i.e.  $(0 \cdot b) \cdot c = 0 \cdot (b \cdot c)$  is obvious by (1).

(b)  $\vdash P(x) \Rightarrow P(S(x))$ , because

$$\begin{aligned} (S(x) \cdot b) \cdot c &= ((x \cdot b) + b) \cdot c \\ &= ((x \cdot b) \cdot c) + (b \cdot c) \quad [\text{Distributive law}] \\ &= (x \cdot (b \cdot c)) + (b \cdot c) \quad [\text{Hypothesis}] \\ &= S(x) \cdot (b \cdot c) \quad [\text{Definition of } \cdot] \end{aligned}$$

(6) Assume  $\neg(a = 0 \vee b = 0)$ , i.e.

$$a \neq 0 \wedge b \neq 0.$$

Since  $a = S(c) = c + 1$  and  $b = S(d) = d + 1$  for some  $c, d \in \mathbb{N}$ ,

$$\begin{aligned} a \cdot b &= (c + 1) \cdot (d + 1) \\ &= c \cdot (d + 1) + (d + 1) \\ &= (c \cdot (d + 1) + d) + 1 \\ &= S(c \cdot (d + 1) + d) \neq 0 \quad [\text{公理 P4}] \end{aligned}$$

$$\neg(a = 0 \vee b = 0) \supset \neg(a \cdot b = 0).$$

Therefore,  $\vdash (a \cdot b = 0) \supset (a = 0 \vee b = 0)$ .

(7) If  $a \cdot b = 1$ , then  $a \neq 0$ . Assume  $a \neq 1$ .

$$a = S(c) = c + 1 \quad \text{for some } c \text{ such that } c \neq 0.$$

Then

$$a \cdot b = (c + 1) \cdot b = c \cdot b + 1 \cdot b = c \cdot b + b = 1$$

$$\therefore c \cdot b = 1 \quad \text{or} \quad b = 1 \quad \text{by (8) .}$$

(a) If  $c \cdot b = 1$ , then  $1 + b = 1$

$$\therefore b = 0 \quad \text{by (6)}$$

$$\therefore a \cdot b = a \cdot 0 = 0 \quad \text{This contradicts to the assumption.}$$

(b) If  $b = 1$ , then  $a \cdot b = a \cdot 1 = a = 1$ , which contradicts to the assumption.

Therefore  $a = 1$ . Similarly,  $b = 1$ .

(8) Assume  $b \neq c$ . By (9) there exist  $c, d$  such that

$$b = c + d \quad \text{or} \quad c = b + d'.$$

If  $b = c + d$ ,  $a \cdot (c + d) = a \cdot c$

$$\therefore a \cdot c + a \cdot d = a \cdot c$$

$$\therefore a \cdot d = 0 \quad \text{by (6)}$$

$$\therefore a = 0 \quad \text{or} \quad d = 0 \quad \text{by (6)}$$

Since  $a \neq 0$  by assumption,  $d = 0$

$\therefore b = c$ , which contradicts to  $b \neq c$ . Similarly, if  $c = b + d'$  then  $b = c$ .

(9)

$$a \cdot b = a \Rightarrow a \cdot b = a \cdot 1$$

$$\Rightarrow b = 1 \quad \text{by (8).}$$

□

## 4.2.8 Ordinals

**DEFINITION 4.2.4.** *Ordered set  $\langle \alpha, \leq \rangle$  is **totally ordered** if*

$$\forall x, y \in \alpha (x \leq y \vee y \leq x).$$

**DEFINITION 4.2.5.** A totally ordered set  $\langle \alpha, \leq \rangle$  is **well-ordered** if

$$\forall \beta \subset \alpha \exists x \in \beta \forall y \in \beta (x \leq y).$$

**DEFINITION 4.2.6.**  $\langle \alpha, \leq \rangle$  is an **ordinal** if

- (1)  $\alpha$  is well-ordered with respect to  $\leq$ , where

$$x \leq y \stackrel{\text{def}}{\iff} (x \in y) \vee (x = y) \quad \text{for } x, y \in \alpha.$$

- (2)  $\beta \in \alpha \implies \beta \subset \alpha$ .

“ $\alpha$  is an ordinal” is denoted by “ $\alpha \in \text{On}$ ”. “On” is not a set, but the proper class of ordinals.

**THEOREM 4.2.9.** The set  $\mathbb{N}$  of natural numbers is an ordinal, which is denoted by  $\omega$ .

**LEMMA 4.2.10.** (1)  $\alpha, \beta \in \text{On} \implies \alpha \cup \beta, \alpha \cap \beta \in \text{On}$ .

- (2) If  $X$  is a non-empty set of ordinals, then  $\bigcup X, \bigcap X \in \text{On}$ .

- (3) If  $\alpha \in \text{On}$ , then  $S(\alpha) \in \text{On}$ , where  $S(\alpha) = \alpha \cup \{\alpha\}$ .

- (4) If  $\alpha \in \text{On}$ , then  $\forall \lambda \in \text{On} (\lambda \in S(\alpha) \equiv \lambda \leq \alpha)$ .

Proved straightforward.

**THEOREM 4.2.11** (Transfinite recursion on On). If  $\forall x, s \exists! y \varphi(x, s, y)$ , define  $G(x, s)$  to be the unique  $y$  such that  $\varphi(x, s, y)$ . Then we can write a formula  $\psi$  for which the following are provable:

- (1)  $\forall x \exists! y \psi(x, y)$ , so  $\psi$  defines a function  $F$ , where  $F(x)$  is the  $y$  such that  $\psi(x, y)$ .
- (2)  $\forall \alpha \in \text{On} (F(\alpha) = G(\alpha, F(\alpha)))$ .

The proof is omitted.

### 4.2.9 Integer

A relation  $\equiv$  on the set  $\mathbb{N} \times \mathbb{N} (= \{\langle a, b \rangle \mid a, b \in \mathbb{N}\})$  defined by

$$\langle a, b \rangle \equiv \langle c, d \rangle \stackrel{\text{def}}{\iff} a + c = b + d$$

is an equivalence relation on  $\mathbb{N} \times \mathbb{N}$ .

*Proof.*

$$\begin{aligned} \vdash a + b &= a + b & \therefore \vdash \langle a, b \rangle &\equiv \langle a, b \rangle \\ \vdash \langle a, b \rangle &\equiv \langle c, d \rangle & \Rightarrow a + d &= b + c \\ & & \Rightarrow c + b &= d + a \\ & & \Rightarrow \langle c, d \rangle &= \langle a, b \rangle \\ \vdash \langle a, b \rangle &\equiv \langle c, d \rangle \wedge \langle c, d \rangle &\equiv \langle e, f \rangle \\ & & \Rightarrow a + d &= b + c \wedge c + f = d + e \\ & & \Rightarrow \vdash a + f &= b + e \\ & & \Rightarrow \vdash \langle a, b \rangle &\equiv \langle e, f \rangle \end{aligned}$$

□

**DEFINITION 4.2.7.** *The quotient of  $\mathbb{N} \times \mathbb{N} / \equiv$  is the set of **integers** denoted by  $\mathbb{Z}$ :*

$$\mathbb{Z} \stackrel{\text{def}}{=} (\mathbb{N} \times \mathbb{N}) / \equiv (= \{|a| \mid a \in \mathbb{N} \times \mathbb{N}\}).$$

*Operations  $+$  and  $\cdot$  on  $\mathbb{Z}$  are defined by*

$$\begin{aligned} \langle a, b \rangle + \langle c, d \rangle &\stackrel{\text{def}}{=} \langle a + c, b + d \rangle, \\ \langle a, b \rangle \cdot \langle c, d \rangle &\stackrel{\text{def}}{=} \langle a \cdot c + b \cdot d, a \cdot d + b \cdot c \rangle. \end{aligned}$$

$+$  and  $\cdot$  are definable on  $\mathbb{Z}$ , i.e.

$$\begin{aligned} \vdash \langle a, b \rangle &\equiv \langle a', b' \rangle \wedge \langle c, d \rangle \equiv \langle c', d' \rangle \Rightarrow \langle a, b \rangle + \langle c, d \rangle \equiv \langle a', b' \rangle + \langle c', d' \rangle \\ \vdash \langle a, b \rangle &\equiv \langle a', b' \rangle \wedge \langle c, d \rangle \equiv \langle c', d' \rangle \Rightarrow \langle a, b \rangle \cdot \langle c, d \rangle \equiv \langle a', b' \rangle \cdot \langle c', d' \rangle \end{aligned}$$

The proof is omitted. In the following, we write  $ab$  instead of  $a \cdot b$  omitting  $\cdot$ .

### 4.2.10 Rational number

A relation  $\equiv$  on the set of pairs of integers,  $\mathbb{Z} \times \mathbb{Z}$ ,

$$\langle a, b \rangle \equiv \langle c, d \rangle \stackrel{\text{def}}{\iff} ad = bc$$

is an equivalence relation.

*Proof.*  $\vdash \langle a, b \rangle \equiv \langle a, b \rangle$  is obvious.

If  $\vdash \langle a, b \rangle \equiv \langle c, d \rangle$ , then  $\vdash ad = bc$ .

$$\therefore \vdash cb = ad \quad \therefore \vdash \langle c, d \rangle \equiv \langle a, b \rangle.$$

If  $\vdash \langle a, b \rangle \equiv \langle c, d \rangle \wedge \langle c, d \rangle \equiv \langle e, f \rangle$ , then

$$\vdash ad = bc \wedge cf = de, \quad \therefore \vdash af = be.$$

$$\therefore \vdash \langle a, e \rangle \equiv \langle b, f \rangle.$$

□

The quotient of  $\mathbb{Z} \times \mathbb{Z}$  by  $\equiv$  is the set of rational numbers, denoted by  $\mathbb{Q}$ .

$$\mathbb{Q} \stackrel{\text{def}}{=} \mathbb{Z} \times \mathbb{Z} / \equiv$$

**DEFINITION 4.2.8.** Operations  $+$  and  $\cdot$  on  $\mathbb{Q}$  are defined by

$$\langle a, b \rangle + \langle c, d \rangle \stackrel{\text{def}}{=} \langle ad + bc, bd \rangle,$$

$$\langle a, b \rangle \cdot \langle c, d \rangle \stackrel{\text{def}}{=} \langle ac, bd \rangle.$$

$+$  and  $\cdot$  are definable, i.e.

$$\vdash \langle a, b \rangle \equiv \langle a', b' \rangle \wedge \langle c, d \rangle \equiv \langle c', d' \rangle \Rightarrow \langle a, b \rangle + \langle c, d \rangle \equiv \langle a', b' \rangle + \langle c', d' \rangle$$

$$\vdash \langle a, b \rangle \equiv \langle a', b' \rangle \wedge \langle c, d \rangle \equiv \langle c', d' \rangle \Rightarrow \langle a, b \rangle \cdot \langle c, d \rangle \equiv \langle a', b' \rangle \cdot \langle c', d' \rangle$$

The proof is omitted.

### 4.2.11 Real number

A real number is defined as a Dedekind cut of rational numbers, that is, a real number  $u$  is the pair of subsets of  $\mathbb{Q}$ , lower part  $L_u$  and upper part  $U_u$ :

$$u = \langle L_u, U_u \rangle \text{ is a \textbf{real number} } \stackrel{\text{def}}{\iff} (D1)_u \wedge \cdots \wedge (D4)_u \text{ where}$$

$$(D1)_u \quad (U_u \subset \mathbb{Q}) \wedge (L_u \subset \mathbb{Q}), \quad (U_u \text{ is the upper part and } L_u \text{ is the lower part})$$

$$(D2)_u \quad \exists x(x \in U_u) \wedge \exists x(x \in \mathbb{Q} \wedge \neg(x \in U_u)),$$

$$(D3)_u \quad \forall x(x \in U_u \supset \forall y(y \in \mathbb{Q} \wedge x \not\leq y \supset y \in U_u)),$$

$$(D4)_u \quad L_u = \{x \in \mathbb{Q} \mid \neg(x \in U_u)\}.$$

The set of all real numbers is denoted by  $\mathbb{R}$ :

$$\mathbb{R} \stackrel{\text{def}}{=} \{\langle L_u, U_u \rangle \in \mathbb{Q} \times \mathbb{Q} \mid (D1)_u \wedge \cdots \wedge (D4)_u\}.$$

If  $u = \langle L_u, U_u \rangle$  and  $v = \langle L_v, U_v \rangle$  are real numbers, then

$$u + v \stackrel{\text{def}}{=} \langle L_{u+v}, U_{u+v} \rangle, \text{ where}$$

$$U_{u+v} \stackrel{\text{def}}{=} \{\mathbb{Q} \mid \forall s \in \mathbb{Q}(x \not\leq s \supset \exists s_1, s_2 \in \mathbb{Q}(s = s_1 + s_2 \wedge s_1 \in U_u \wedge s_2 \in U_v))\},$$

$$L_{u+v} \stackrel{\text{def}}{=} \{x \in \mathbb{Q} \mid \neg(x \in U_{u+v})\}.$$

$$u \leq v \stackrel{\text{def}}{\iff} L_u \subset L_v.$$

If  $u, v \geq 0$ ,

$$uv \stackrel{\text{def}}{=} \langle L_{uv}, U_{uv} \rangle, \text{ where}$$

$$U_{uv} \stackrel{\text{def}}{=} \{\mathbb{Q} \mid \forall s \in \mathbb{Q}(x \not\leq s \supset \exists s_1, s_2 \in \mathbb{Q}(s = s_1 s_2 \wedge s_1 \in U_u \wedge s_2 \in U_v))\},$$

$$L_{uv} = \{x \in \mathbb{Q} \mid \neg(x \in U_{uv})\}.$$

Then  $\langle L_{u+v}, U_{u+v} \rangle$  and  $\langle L_{uv}, U_{uv} \rangle$  satisfy the conditions of real number, i.e.  $u + v, uv \in \mathbb{R}$ . The definition of  $uv$  can be extended to all real numbers  $u, v \in \mathbb{R}$ .

Real number  $u = \langle L_u, U_u \rangle$  is determined by the lower part  $L_u$  or upper part  $U_u$ . So we sometimes use the lower part  $L_u$  or upper part  $U_u$  to denote  $u$ .

### 4.2.12 Complex number

Complex number is defined as a pair of real numbers. The set of all complex numbers is denoted by  $\mathbb{C}$ :

$$\mathbb{C} \stackrel{\text{def}}{=} \mathbb{R} \times \mathbb{R}$$

For  $\langle a, b \rangle, \langle c, d \rangle \in \mathbb{C}$ ,

$$\langle a, b \rangle = \langle c, d \rangle \stackrel{\text{def}}{\iff} (a = c) \wedge (b = d).$$

$$\langle a, b \rangle + \langle c, d \rangle \stackrel{\text{def}}{\iff} \langle a + c, b + d \rangle,$$

$$\langle a, b \rangle \cdot \langle c, d \rangle \stackrel{\text{def}}{\iff} \langle ac - bd, ad + bc \rangle.$$

We identify  $\langle a, 0 \rangle$  with real number  $a$ , and denote  $\langle 0, 1 \rangle$  by  $i$ .

### 4.2.13 Universe of ZFC

We have defined the class On of ordinals in ZFC. A universe  $V$  of ZFC will be defined in ZFC using transfinite recursion on On.

First we define a function  $\alpha \mapsto V_\alpha$  that assigns the set  $V_\alpha$  to ordinal  $\alpha$ , as follows:

- (1)  $V_0 = \emptyset$
- (2)  $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
- (3)  $V_\alpha = \bigcup_{\beta \in \alpha} V_\beta$ , whenever  $\alpha$  is a limit ordinal  
 $V = \bigcup_{\alpha \in \text{On}} V_\alpha$ .

The Power Set axiom is used to obtain  $V_{\alpha+1}$  from  $V_\alpha$ . Replacement and Union allow one to form  $V_\alpha$  for a limit ordinal  $\alpha$ . The axiom of Infinity is necessary to prove the existence of  $\omega$  and transfinite sequence of ordinals. Finally, ZFC proves that every set belongs to some  $V_\alpha$ . For  $u, v \in V$ ,

$$\text{truth value of } u = v \text{ is } \begin{cases} 1, & u = v \\ 0, & u \neq v \end{cases}$$

The proper class  $V$ , together with the relations  $\in$  and  $=$ , satisfies all axioms of ZFC. Thus,  $V$  is a universe of ZFC.

This is rephrased as follows.

In ZFC, a subset  $u$  of  $v$  is represented as the characteristic function  $\chi_u : v \rightarrow \mathbf{2}$ . Thus, the **universe**  $V$  of ZFC is defined inductively, as follows.

$$\begin{aligned} V_\alpha &= \{u \mid \exists \beta < \alpha \exists \mathcal{D}u \subset V_\beta (u : \mathcal{D}u \rightarrow \mathbf{2})\}, \\ V &= \bigcup_{\alpha \in On} V_\alpha. \end{aligned}$$

The least  $\alpha$  such that  $u \in V_\alpha$  is called the **rank** of  $u$ .

Truth values of atomic formulas on the universe  $V$  are given as

$$\begin{aligned} \llbracket u=v \rrbracket &= \bigwedge_{x \in \mathcal{D}u} (u(x) \supset \llbracket x \in v \rrbracket) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \supset \llbracket x \in u \rrbracket), \\ \llbracket u \in v \rrbracket &= \bigvee_{x \in \mathcal{D}v} (v(x) \wedge \llbracket u=x \rrbracket), \end{aligned}$$

where  $\wedge, \vee, \neg, \forall, \exists$  and  $\supset$  are operators on the Boolean algebra  $\mathbf{2}$ .

Logical operators  $\wedge, \vee, \neg, \forall, \exists$  and  $\supset$  represent algebraic operators on the Boolean algebra.

$$\begin{aligned} \llbracket \varphi \wedge \psi \rrbracket &= \llbracket \varphi \rrbracket \wedge \llbracket \psi \rrbracket \\ \llbracket \varphi \vee \psi \rrbracket &= \llbracket \varphi \rrbracket \vee \llbracket \psi \rrbracket \\ \llbracket \neg \varphi \rrbracket &= \neg \llbracket \varphi \rrbracket \\ \llbracket \forall x \varphi(x) \rrbracket &= \bigwedge_{x \in V} \llbracket \varphi(x) \rrbracket \\ \llbracket \exists x \varphi(x) \rrbracket &= \bigvee_{x \in V} \llbracket \varphi(x) \rrbracket \\ \llbracket \varphi \supset \psi \rrbracket &= \neg \llbracket \varphi \rrbracket \vee \llbracket \psi \rrbracket \end{aligned}$$

Then every formula has truth value 1 or 0 on  $V$ .

If a formula  $\varphi$  is true in  $V$ , that is,  $\llbracket \varphi \rrbracket = 1$  in  $V$ , then we say  $\varphi$  is valid in  $V$ , and write

$$V \models \varphi.$$

**THEOREM 4.2.12.** *The set-theoretical axioms of ZFC, A1-A9 and AC, are all true in the universe  $V$ , i.e. each axiom is valid in  $V$ . Hence*

$$\text{if } \text{ZFC} \vdash \varphi, \text{ then } V \models \varphi.$$

# Chapter 5

## Various global logics

### 5.1 Global logic

Sentence “ $\dots$  is true ” is a metalogical statement for classical logical. Global logic is obtained by introducing formulas of the form:

$$\varphi \rightarrow \psi(\text{“}\varphi \text{ implies } \psi\text{”}) \quad \text{or} \quad \Box\varphi(\text{“}\varphi \text{ is true ”}).$$

Logical operation  $\rightarrow$ , which corresponds to lattice operation  $\rightarrow$ , was introduced in Titani [13], and called **basic implication**.

Corresponding lattice operation  $\rightarrow$  is defined by

$$(a \rightarrow b) = \begin{cases} 1 & \text{if } a \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Globalization  $\Box$  corresponds to lattice operation defined by

$$\Box a = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The globalization  $\Box$  is defined by using the basic implication:

$$\Box\varphi \stackrel{\text{def}}{\iff} (\varphi \rightarrow \varphi) \rightarrow \varphi.$$

If a logical system has an implication  $\supset$ , then the basic implication  $\rightarrow$  is defined by using globalization:

$$\varphi \rightarrow \psi \stackrel{\text{def}}{\iff} \Box(\varphi \supset \psi).$$

**DEFINITION 5.1.1.** *A logic with logical operation  $\rightarrow$  or  $\Box$  is called a **global logic**.*

Logical system obtained from LK by adding  $\rightarrow$  or  $\Box$  is **global classical logic** denoted by GLK, and logical system obtained from LJ by adding  $\rightarrow$  or  $\Box$  is **global intuitionistic logic** denote by GLJ.

Formulas of the form  $\varphi \rightarrow \psi$  or  $\Box\varphi$  are interpreted as either true or false, where truth value of  $\varphi$  is denoted by  $\llbracket\varphi\rrbracket$ .

$$\llbracket\varphi \rightarrow \psi\rrbracket = \begin{cases} 1, & \text{if } \llbracket\varphi\rrbracket \leq \llbracket\psi\rrbracket \\ 0, & \text{otherwise} \end{cases} \quad \llbracket\Box\varphi\rrbracket = \begin{cases} 1, & \varphi \text{ is true} \\ 0, & \varphi \text{ is not true} \end{cases}$$

That is, the formulas of the form  $\Box\varphi$  or  $\varphi \rightarrow \psi$  follow the rules of classical logic LK.

## 5.2 Logical system of global logic

Logical systems consist of alphabet and inference rules.

### Alphabet of logical systems

- (1) Constants :  $c, c_0, c_1, c_2, \dots$ ,
- (2) Free variables :  $a, a_0, a_1, a_2, \dots$ ,
- (3) Bound variables :  $x, x_0, x_1, x_2, \dots$ ,
- (4) Predicate constants with  $n$  arguments :  $p^n, p_0^n, p_1^n, p_2^n, \dots$ ,  $(n \geq 0)$ ,
- (5) Logical symbols :  $\wedge, \vee, \forall, \exists, \rightarrow$  and some proper symbols for each system.
- (6) Auxiliary symbols :  $( )$ , and commas.

### Terms

Individual constants and free variables are called **terms**, and they are denoted by  $t_1, t_2, \dots$ .

## Formulas

If  $p_i^n$  is a predicate constant with  $n$  arguments and  $t_1, \dots, t_n$  are terms, then  $p_i^n(t_{i_1}, \dots, t_{i_n})$  is called a **primitive formula**. **Formulas** are constructed from the primitive formulas using logical symbols, as follows.

- (1) The primitive formulas are formulas.
- (2) If  $\varphi$  and  $\psi$  are formulas, then  $(\varphi \wedge \psi)$ ,  $(\varphi \vee \psi)$  are formulas.
- (3) If  $\varphi(a)$  is a formula with free variable  $a$ , and  $x$  is a bound variable which does not occur in  $\varphi(a)$ , then  $\forall x\varphi(x)$  and  $\exists x\varphi(x)$  are formulas, where  $\varphi(x)$  is obtained from  $\varphi(a)$  by substituting  $x$  for all  $a$  in  $\varphi(a)$ .
- (4) For a global logic, if  $\varphi$  and  $\psi$  are formulas then  $\varphi \rightarrow \psi$  is a formula.

A formula without any occurrence of free variables is called a **sentence**. A formula which appears in the construction of a formula is called a **subformula** of the formula.

Formulas are denoted by  $\varphi, \psi, \dots; \varphi(a), \psi(a), \dots$ .

**DEFINITION 5.2.1.** *A sequent is a formal expression of the form*

$$\varphi_1, \dots, \varphi_m \Rightarrow \psi_1, \dots, \psi_n.$$

The part " $\varphi_1, \dots, \varphi_m$ " is the **antecedent**, and " $\psi_1, \dots, \psi_n$ " the **succedent** of the sequent.

Finite sequences of formulas are denoted by  $\Gamma, \Delta, \dots$ . So sequents are written as the form  $\Gamma \Rightarrow \Delta$ .

An **inference** is an expression of the form

$$\frac{S_1}{S} \quad \text{or} \quad \frac{S_1 \quad S_2}{S},$$

where  $S_1, S_2$  and  $S$  are sequents.  $S_1$  and  $S_2$  are called the **upper sequents** and  $S$  is called the **lower sequent** of the inference.

### 5.3 Interpretation of logical systems

A **model**  $M$  of a logic is a triple  $M = \langle \mathcal{L}, D, I \rangle$ , where

- $\mathcal{L}$  is a lattice of truth values of the logic, on which logical operations are interpreted as algebraic operations on  $\mathcal{L}$ .
- $D$  is a domain of variables, and
- $I$  is an interpretation of predicate symbols

$$I(p^n) : D^n \rightarrow \mathcal{L}.$$

A **D-assignment**  $v$  is a mapping

$$v : FV \rightarrow D,$$

where  $FV$  is the set of all individual free variables.

For a D-assignment  $v$  and  $d \in D$ ,  $v(d/a)$  denotes the D-assignment :  $FV \rightarrow D$  such that for  $x \in FV$

$$v'(d/a)(x) = \begin{cases} v(x) & x \neq a \\ v(d) & x = a \end{cases}$$

Let  $M = \langle \mathcal{L}, D, I \rangle$  be a model and  $v$  be a D-assignment. The truth value of formula  $\varphi$  with respect to  $M$  and  $v$  is denoted by  $\varphi[M, v] \in \mathcal{L}$ . If  $M$  and  $v$  are omissible, then we denote  $\varphi[M, v]$  by  $\llbracket \varphi \rrbracket$ .

The logic are interpreted on the model:

$$\begin{aligned}
 \llbracket \varphi \wedge \psi \rrbracket &= \llbracket \varphi \rrbracket \wedge \llbracket \psi \rrbracket \\
 \llbracket \varphi \vee \psi \rrbracket &= \llbracket \varphi \rrbracket \vee \llbracket \psi \rrbracket \\
 \llbracket \forall x \varphi(x) \rrbracket &= \bigwedge_{x \in D} \llbracket \varphi(x) \rrbracket \\
 \llbracket \exists x \varphi(x) \rrbracket &= \bigvee_{x \in D} \llbracket \varphi(x) \rrbracket \\
 \llbracket \varphi \rightarrow \psi \rrbracket &= \begin{cases} 1, & \llbracket \varphi \rrbracket \leq \llbracket \psi \rrbracket \\ 0, & \text{otherwise} \end{cases} \\
 \llbracket \Box \varphi \rrbracket &= \Box \llbracket \varphi \rrbracket = \begin{cases} 1, & \llbracket \varphi \rrbracket = 1 \\ 0, & \text{otherwise} \end{cases} \\
 \llbracket \Phi(\varphi_1, \dots, \varphi_n) \rrbracket &= \llbracket \Phi \rrbracket(\llbracket \varphi_1 \rrbracket, \dots, \llbracket \varphi_n \rrbracket) \quad \text{for other logical operator } \Phi \\
 &\quad \text{and corresponding operator } \llbracket \Phi \rrbracket \text{ on } \mathcal{L}.
 \end{aligned}$$

If  $\Phi$  and  $\Psi$  are not necessarily finite set of formulas, then  $\Phi \Rightarrow \Psi$  is called a **generalized sequent**.  $\Phi \Rightarrow \Psi$  is said to be **valid** in model  $M = \langle \mathcal{L}, D, I \rangle$  and D-assignment  $v$ , in symbols,

$$[M, v] \models \Phi \Rightarrow \Psi,$$

if

$$\bigwedge \{ \varphi[M, v] \mid \varphi \in \Phi \} \leq \bigvee \{ \psi[M, v] \mid \psi \in \Psi \}.$$

Or

$$\bigwedge \{ \llbracket \varphi \rrbracket \mid \varphi \in \Phi \} \leq \bigvee \{ \llbracket \psi \rrbracket \mid \psi \in \Psi \} \quad \text{on model } M \text{ and } v.$$

If  $[M, v] \models \Phi \Rightarrow \Psi$  for every D-assignment  $v$ , then we write as

$$M \models \Phi \Rightarrow \Psi.$$

If the generalized sequent  $\Phi \Rightarrow \Psi$  is valid in every model  $M = \langle \mathcal{L}, D, I \rangle$  and every D-assignment  $v$ , then we say that  $\Phi \Rightarrow \Psi$  is **valid**, and write as

$$\models \Phi \Rightarrow \Psi.$$

“ $\Rightarrow$ ” in sequents of LK is not a logical operation, but a relation.

We will adopt the classical 2-valued logic as the meta-logic, which is the underlying basic logic. The classical 2-valued logic is represented by the Boolean algebra  $\mathbf{2}$  consisting of 1 and 0, which is a sub-algebra of every complete lattice. Thus, the lattice order  $\leq$  can be considered as an operation:

$$\mathcal{L} \times \cdots \times \mathcal{L} \rightarrow \mathbf{2} \quad \text{on a lattice } \mathcal{L}.$$

We introduced in Titani [13] a basic implication  $\rightarrow$  on the lattice order:

$$(a \rightarrow b) = \begin{cases} 1 & a \leq b \\ 0 & a \not\leq b. \end{cases}$$

## 5.4 Lattice valued logic LL

**Lattice valued logic** LL is a logical system which is a counterpart of complete lattice. LL is a global logic with basic implication  $\rightarrow$ , which was introduced in Titani [13].

### 5.4.1 Formal system of lattice valued logic LL

The primitive symbols of operations of LL are

$$\wedge, \vee, \neg, \rightarrow, \forall, \exists.$$

The following symbols are defined from the primitive symbols.

$$\begin{aligned} \top &\stackrel{\text{def}}{\iff} (\varphi \rightarrow \varphi) \\ \perp &\stackrel{\text{def}}{\iff} \neg \top \\ \neg \varphi &\stackrel{\text{def}}{\iff} (\varphi \rightarrow \perp) \\ \varphi \leftrightarrow \psi &\stackrel{\text{def}}{\iff} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \\ \Box \varphi &\stackrel{\text{def}}{\iff} (\varphi \rightarrow \varphi) \rightarrow \varphi \\ \Diamond \varphi &\stackrel{\text{def}}{\iff} \neg(\neg \Box \varphi) \end{aligned}$$

### 5.4.2 Inference rules of LL

First of all, we define  $\Box$ -closed formulas inductively, as follows :

- (1) Formulas of the form  $(\varphi \rightarrow \psi)$  are  $\Box$ -closed ;
- (2) If formulas  $\varphi$  and  $\psi$  are  $\Box$ -closed, then  $\varphi \wedge \psi$  and  $\varphi \vee \psi$  are also  $\Box$ -closed ;
- (3) If a formula  $\varphi(a)$  is a  $\Box$ -closed formula with free variable  $a$ , then  $\forall x\varphi(x)$  and  $\exists x\varphi(x)$  are also  $\Box$ -closed;
- (4)  $\Box$ -closed formulas of LL are only those obtained by (1)–(3).

$\Gamma, \Delta, \Pi, \Lambda, \dots$  will be used to denote finite sequences of formulas ;  $\overline{\varphi}, \overline{\psi}, \dots$  to denote  $\Box$ -closed formulas ; and  $\overline{\Gamma}, \overline{\Delta}, \overline{\Pi}, \overline{\Lambda}, \dots$  to denote finite sequences of  $\Box$ -closed formulas.

**Begining sequents:** Every proof of LL starts with **logical axioms** which are sequents of the form  $\varphi \Rightarrow \varphi$ .

**Structural rules :**

$$\begin{array}{ll}
 \text{Thinning :} & \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \\
 \\
 \text{Contraction :} & \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \\
 \\
 \text{Interchange :} & \frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda} \\
 \\
 \text{Cut :} & \frac{\Gamma \Rightarrow \overline{\Delta}, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \overline{\Delta}, \Lambda} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \overline{\Pi} \Rightarrow \Lambda}{\Gamma, \overline{\Pi} \Rightarrow \Delta, \Lambda} \\
 \\
 & \frac{\Gamma \Rightarrow \Delta, \overline{\varphi} \quad \overline{\varphi}, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}
 \end{array}$$

**Logical rules:**

$$\neg : \quad \frac{\Gamma \Rightarrow \bar{\Delta}, A}{\neg A, \Gamma \Rightarrow \bar{\Delta}} \quad \frac{\Gamma \Rightarrow \Delta, \bar{A}}{\neg \bar{A}, \Gamma \Rightarrow \Delta} \quad \frac{A, \bar{\Gamma} \Rightarrow \bar{\Delta}}{\bar{\Gamma} \Rightarrow \bar{\Delta}, \neg A} \quad \frac{\bar{A}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \bar{A}},$$

$$\wedge : \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \bar{\Delta}, \varphi \quad \Gamma \Rightarrow \bar{\Delta}, \psi}{\Gamma \Rightarrow \bar{\Delta}, \varphi \wedge \psi}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \bar{\varphi} \quad \Gamma \Rightarrow \Delta, \bar{\psi}}{\Gamma \Rightarrow \Delta, \bar{\varphi} \wedge \bar{\psi}}$$

$$\vee : \quad \frac{\varphi, \bar{\Gamma} \Rightarrow \Delta \quad \psi, \bar{\Gamma} \Rightarrow \Delta}{\varphi \vee \psi, \bar{\Gamma} \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\frac{\bar{\varphi}, \Gamma \Rightarrow \Delta \quad \bar{\psi}, \Gamma \Rightarrow \Delta}{\bar{\varphi} \vee \bar{\psi}, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\rightarrow : \quad \frac{\Gamma \Rightarrow \bar{\Delta}, \varphi \quad \psi, \bar{\Pi} \Rightarrow \Lambda}{(\varphi \rightarrow \psi), \Gamma, \bar{\Pi} \Rightarrow \bar{\Delta}, \Lambda} \quad \frac{\varphi, \bar{\Gamma} \Rightarrow \bar{\Delta}, \psi}{\bar{\Gamma} \Rightarrow \bar{\Delta}, (\varphi \rightarrow \psi)} \quad \frac{\bar{\varphi}, \Gamma \Rightarrow \Delta, \bar{\psi}}{\Gamma \Rightarrow \Delta, (\bar{\varphi} \rightarrow \bar{\psi})}$$

$$\forall : \quad \frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \bar{\Delta}, \varphi(a)}{\Gamma \Rightarrow \bar{\Delta}, \forall x \varphi(x)} \quad \frac{\Gamma \Rightarrow \Delta, \bar{\varphi}(a)}{\Gamma \Rightarrow \Delta, \forall x \bar{\varphi}(x)}$$

where  $t$  is any term      where  $a$  is a free variable which does not occur in the lower sequent.

$$\exists : \quad \frac{\varphi(a), \bar{\Gamma} \Rightarrow \Delta}{\exists x \varphi(x), \bar{\Gamma} \Rightarrow \Delta} \quad \frac{\bar{\varphi}(a), \Gamma \Rightarrow \Delta}{\exists x \bar{\varphi}(x), \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)}$$

where  $a$  is a free variable which does not occur in the lower sequent.      where  $t$  is any term

### 5.4.3 LL-provability

**DEFINITION 5.4.1.** *Each logical axiom  $\varphi \Rightarrow \varphi$  is provable in LL, and if up-persequent(s) of inference rules are provable in LL, then the lower sequent is*

provable in LL. A tree of provable formulas is called a **proof** of the lowermost sequent.

“A sequent  $\Gamma \Rightarrow \Delta$  is provable in LL ” is expressed by

$$\text{LL} \vdash \Gamma \Rightarrow \Delta.$$

**DEFINITION 5.4.2.** Generalized sequent  $\Phi \Rightarrow \Psi$  is said to be **LL-provable**, if there exist finite subsequences  $\Gamma \subset \Phi$  and  $\Delta \subset \Psi$  such that

$$\text{LL} \vdash \Gamma \Rightarrow \Delta.$$

**DEFINITION 5.4.3.** Formulas  $\varphi \wedge \psi$  in the rule  $(\wedge)$ ,  $\varphi \vee \psi$  in the rule  $(\vee)$ ,  $\forall x\varphi(x)$  in the rule  $(\forall)$  and  $\exists x\varphi(x)$  in the rule  $(\exists)$  are called the **principal formulas**.

**THEOREM 5.4.1.**

- (1)  $\text{LL} \vdash \varphi \wedge \psi, \Gamma \Rightarrow \Delta$  if and only if  $\text{LL} \vdash \varphi, \psi, \Gamma \Rightarrow \Delta$
- (2)  $\text{LL} \vdash \Gamma \Rightarrow \Delta, \varphi \vee \psi$  if and only if  $\text{LL} \vdash \Gamma \Rightarrow \Delta, \varphi, \psi$
- (3)  $\text{LL} \vdash (\varphi_1 \wedge \cdots \wedge \varphi_n) \Rightarrow \varphi_i \quad (i = 1, \dots, n)$
- (4)  $\text{LL} \vdash \psi_j \Rightarrow (\psi_1 \vee \cdots \vee \psi_n) \quad (j = 1, \dots, n)$
- (5)  $\text{LL} \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \psi') \Rightarrow \varphi \wedge (\psi \vee \psi')$
- (6)  $\text{LL} \vdash \varphi \vee (\psi \wedge \psi') \Rightarrow (\varphi \vee \psi) \wedge (\varphi \vee \psi')$
- (7)  $\text{LL} \vdash \exists x(\varphi \wedge \psi(x)) \Rightarrow (\varphi \wedge \exists x\psi(x))$
- (8)  $\text{LL} \vdash (\varphi \vee \forall x\psi(x)) \Rightarrow \forall x(\varphi \vee \psi(x))$
- (9) If  $A(\varphi)$  is a formula with subformula  $\varphi$  and  $\text{LL} \vdash \varphi \Leftrightarrow \psi$ , then

$$\text{LL} \vdash A(\varphi) \Leftrightarrow A(\psi).$$

*Proof.* (1)  $\text{LL} \vdash \varphi, \psi \Rightarrow \varphi \wedge \psi$  by  $\wedge$ -right :

$$\frac{\frac{\varphi \Rightarrow \varphi}{\psi, \varphi \Rightarrow \varphi} \text{ (Thinning )} \quad \frac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi} \text{ (Thinning )}}{\frac{\varphi, \psi \Rightarrow \varphi}{\varphi, \psi \Rightarrow \varphi \wedge \psi} \text{ (Interchange)}} \text{ (}\wedge\text{-right)}$$

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and

$$\frac{\varphi, \psi \Rightarrow \varphi \wedge \psi \quad \varphi \wedge \psi, \Gamma \Rightarrow \Delta}{\varphi, \psi, \Gamma \Rightarrow \Delta} \text{ (Cut)}$$

Hence,  $\text{LL} \vdash \varphi \wedge \psi, \Gamma \Rightarrow \Delta$  implies  $\text{LL} \vdash \varphi, \psi, \Gamma \Rightarrow \Delta$ .

The converse follows from  $\wedge$ -left :

$$\frac{\frac{\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \psi, \Gamma \Rightarrow \Delta} (\wedge\text{-left})}{\psi, \varphi \wedge \psi, \Gamma \Rightarrow \Delta} \text{ (Interchange)}}{\frac{\varphi \wedge \psi, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} (\wedge\text{-left})} \text{ (Contraction)}$$

- (2)  $\text{LL} \vdash \varphi \Rightarrow \varphi, \psi$  and  $\text{LL} \vdash \psi \Rightarrow \varphi, \psi$  by Thinning and Interchange.  
Hence, by using  $\vee$ -left,

$$\text{LL} \vdash \varphi \vee \psi \Rightarrow \varphi, \psi.$$

Then by using Cut,

$$\text{LL} \vdash \Gamma \Rightarrow \Delta, \varphi \vee \psi \text{ implies } \text{LL} \vdash \Gamma \Rightarrow \Delta, \varphi, \psi$$

The converse follows from  $\vee$ -right, in the similar way to (1).

- (3)

$$\frac{\frac{\varphi_i \Rightarrow \varphi_i}{\varphi_1, \dots, \varphi_m \Rightarrow \varphi_i} \text{ (Thinning)}}{\varphi_1 \wedge \dots \wedge \varphi_m \Rightarrow \varphi_i \text{ by (1)}}$$

- (4) Similarly, by Thinning and Rule (2),

$$\text{LL} \vdash \psi_j \Rightarrow (\psi_1 \vee \dots \vee \psi_n) \quad (j = 1, \dots, n).$$

- (5)

$$\frac{\frac{\vdots}{\varphi \wedge \psi \Rightarrow \varphi \wedge (\psi \vee \psi')}}{(\varphi \wedge \psi) \vee (\varphi \wedge \psi) \Rightarrow \varphi \wedge (\psi \vee \psi')}$$

- (6) Similar to (5).

(7)

$$\frac{\frac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi(a) \Rightarrow \varphi} \quad \frac{\vdots}{\varphi \wedge \psi(a) \Rightarrow \exists x\psi(x)}}{\frac{\varphi \wedge \psi(a) \Rightarrow \varphi \wedge \exists x\psi(x)}{\exists x(\varphi \wedge \psi(x)) \Rightarrow \varphi \wedge \exists x\psi(x)}}$$

(8)

$$\frac{\frac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee \psi(a)} \quad \frac{\vdots}{\forall x\psi(x) \Rightarrow \varphi \vee \psi(a)}}{\frac{\varphi \vee \forall x\psi(x) \Rightarrow \varphi \vee \psi(a)}{\varphi \vee \forall x\psi(x) \Rightarrow \forall x(\varphi \vee \psi(x))}}$$

(9)

$$\frac{\frac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi} \quad \frac{\varphi \wedge \psi \Rightarrow \psi \quad \psi \Rightarrow \psi'}{\varphi \wedge \psi \Rightarrow \psi'}}{\varphi \wedge \psi \Rightarrow \varphi \wedge \psi'}$$

Hence, if  $\text{LL} \vdash \psi \Leftrightarrow \psi'$ , then  $\text{LL} \vdash \varphi \wedge \psi \Leftrightarrow \varphi \wedge \psi'$ .

$$\frac{\frac{\psi' \Rightarrow \psi'}{\varphi \wedge \psi' \Rightarrow \psi'} \quad \frac{\varphi \wedge \psi' \Rightarrow \varphi \quad \varphi \Rightarrow \varphi'}{\varphi \wedge \psi' \Rightarrow \varphi'}}{\varphi \wedge \psi' \Rightarrow \varphi' \wedge \psi'}$$

Hence, if  $\text{LL} \vdash \varphi \Leftrightarrow \varphi'$ , and  $\text{LL} \vdash \psi \Leftrightarrow \psi'$  then  $\text{LL} \vdash \varphi \wedge \psi' \Leftrightarrow \varphi' \wedge \psi'$  and  $\text{LL} \vdash \varphi \wedge \psi \Leftrightarrow \varphi \wedge \psi'$ .

Simiraly,

$\text{LL} \vdash \varphi \Leftrightarrow \varphi'$  and  $\text{LL} \vdash \psi \Leftrightarrow \psi'$  implies  $\text{LL} \vdash \varphi \vee \psi \Leftrightarrow \varphi' \vee \psi'$ ,

$\text{LL} \vdash \varphi(a) \Leftrightarrow \varphi'(a)$  implies

$\text{LL} \vdash \forall x\varphi(x) \Leftrightarrow \forall x\varphi'(x)$  and  $\text{LL} \vdash \exists x\varphi(x) \Leftrightarrow \exists x\varphi'(x)$ .

Since  $A(\varphi)$  is constructed from  $\varphi$  by  $\wedge$ ,  $\vee$ ,  $\forall$  and  $\exists$ ,

$\text{L} \vdash \varphi \Leftrightarrow \varphi'$  implies  $\text{LL} \vdash A(\varphi) \Leftrightarrow A(\varphi')$

by induction on the complexity of  $A(\varphi)$ .

□

**THEOREM 5.4.2.**

- (1)  $LL \vdash \varphi \Rightarrow \psi$  *if and only if*  $LL \vdash \Rightarrow (\varphi \rightarrow \psi)$
- (2)  $LL \vdash \varphi, (\varphi \rightarrow \psi) \Rightarrow \psi$
- (3)  $LL \vdash (\varphi \rightarrow \psi), (\psi \rightarrow \theta) \Rightarrow (\varphi \rightarrow \theta)$
- (4)  $LL \vdash \Rightarrow ((\varphi_1 \wedge \cdots \wedge \varphi_m) \rightarrow \varphi_i) \quad (i = 1, \dots, m)$
- (5)  $LL \vdash \Rightarrow (\psi_j \rightarrow (\psi_1 \vee \cdots \vee \psi_n)) \quad (j = 1, \dots, n)$
- (6)  $LL \vdash (\theta \rightarrow \varphi), (\theta \rightarrow \psi) \Rightarrow (\theta \rightarrow (\varphi \wedge \psi))$
- (7)  $LL \vdash (\varphi \rightarrow \theta), (\psi \rightarrow \theta) \Rightarrow ((\varphi \vee \psi) \rightarrow \theta)$
- (8)  $LL \vdash \varphi \wedge \neg \varphi \Rightarrow \theta.$
- (9)  $LL \vdash \varphi, \bar{\Gamma} \Rightarrow \bar{\Delta}, \psi$  *implies*  $LL \vdash \neg \psi, \bar{\Gamma} \Rightarrow \bar{\Delta}, \neg \varphi$
- (10)  $LL \vdash \varphi \Rightarrow \neg \neg \varphi$  ;  $LL \vdash \Box \varphi \Leftrightarrow \neg \neg \Box \varphi$  ;
- (11)  $LL \vdash \neg(\varphi \vee \psi) \Leftrightarrow (\neg \varphi \wedge \neg \psi)$
- (12)  $LL \vdash (\neg \varphi \vee \neg \psi) \Rightarrow \neg(\varphi \wedge \psi)$
- (13)  $LL \vdash \Box \varphi \Rightarrow \varphi$
- (14)  $LL \vdash \varphi \Rightarrow \Diamond \varphi$
- (15)  $LL \vdash \bar{\Gamma} \Rightarrow \bar{\Delta}, \varphi$  *if and only if*  $LL \vdash \bar{\Gamma} \Rightarrow \bar{\Delta}, \Box \varphi$
- (16) *If  $\varphi$  is  $\Box$ -closed, then*  $LL \vdash \varphi \Leftrightarrow \Box \varphi$
- (17)  $LL \vdash \Box \varphi \Leftrightarrow \Box \Box \varphi$
- (18)  $LL \vdash \varphi, \bar{\Gamma} \Rightarrow \bar{\Delta}$  *if and only if*  $LL \vdash \Diamond \varphi, \bar{\Gamma} \Rightarrow \bar{\Delta}$
- (19)  $LL \vdash \Box \varphi \wedge \exists x \psi(x) \Leftrightarrow \exists x(\Box \varphi \wedge \psi(x));$   
 $LL \vdash \varphi \wedge \exists x \Box \psi(x) \Leftrightarrow \exists x(\varphi \wedge \Box \psi(x))$
- (20)  $LL \vdash \forall x \Box \varphi(x) \Leftrightarrow \Box \forall x \varphi(x)$

$$(21) \quad \text{LL} \vdash (\Box\varphi \rightarrow \psi) \Leftrightarrow (\Box\varphi \rightarrow \Box\psi) \Leftrightarrow (\neg\Box\varphi \vee \Box\psi)$$

$$(22) \quad \text{LL} \vdash \Rightarrow \Box\varphi \vee \neg\Box\varphi$$

$$(23) \quad \text{LL} \vdash ((\varphi \wedge \Box\xi) \rightarrow \psi) \Rightarrow ((\neg\psi \wedge \Box\xi) \rightarrow \neg\varphi)$$

$$(24) \quad \text{LL} \vdash \neg\varphi \Leftrightarrow \Box\neg\varphi$$

$$(25) \quad \text{LL} \vdash (\varphi \rightarrow \Box\psi) \Rightarrow (\Diamond\varphi \rightarrow \Box\psi)$$

$$(26) \quad \text{LL} \vdash \Diamond(\Box\varphi \wedge \psi) \Rightarrow \Box\varphi \wedge \Diamond\psi$$

$$(27) \quad \text{LL} \vdash \exists x\Diamond\varphi(x) \Leftrightarrow \Diamond\exists x\varphi(x)$$

*Proof.* (1) By the inference rule ( $\rightarrow$ ).

(2)

$$\frac{\varphi \Rightarrow \varphi \quad \psi \Rightarrow \psi \quad (\text{axioms})}{(\varphi \rightarrow \psi), \varphi \Rightarrow \psi} \rightarrow\text{-left}$$

It follows that

$$\text{LL} \vdash \varphi, (\varphi \rightarrow \psi) \Rightarrow \psi.$$

(3) Since  $(\psi \rightarrow \theta)$  is  $\Box$ -closed, by  $\rightarrow$ -right ,

$$\text{LL} \vdash \varphi, (\varphi \rightarrow \psi), (\psi \rightarrow \theta) \Rightarrow \psi, (\psi \rightarrow \theta)$$

and

$$\text{LL} \vdash \psi, (\psi \rightarrow \theta) \Rightarrow \theta.$$

Hence, by rule Cut,

$$\text{LL} \vdash \varphi, (\varphi \rightarrow \psi), (\psi \rightarrow \theta) \Rightarrow \theta.$$

(4)

$$\frac{\frac{\frac{\varphi_i \Rightarrow \varphi_i}{\varphi_1, \dots, \varphi_m \Rightarrow \varphi_i} (\text{Thinning})}{\varphi_1 \wedge \dots \wedge \varphi_m \Rightarrow \varphi_i} (\text{by Rule (1) })}{\Rightarrow ((\varphi_1 \wedge \dots \wedge \varphi_m) \rightarrow \varphi_i)} \rightarrow\text{-right}$$

Hence,

$$\text{LL} \vdash \Rightarrow ((\varphi_1 \wedge \dots \wedge \varphi_m) \rightarrow \varphi_i).$$

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(5) By Thinning and Rule (2),

$$LL \vdash \Rightarrow (\psi_j \rightarrow (\psi_1 \vee \cdots \vee \psi_n)) \quad (j = 1, \dots, n)$$

in the similar way as (6).

(6) By using (4) and Rules  $\wedge$ -right,  $\rightarrow$ -right,

$$\frac{\frac{\theta, (\theta \rightarrow \varphi), (\theta \rightarrow \psi) \Rightarrow \varphi \quad \theta, (\theta \rightarrow \varphi), (\theta \rightarrow \psi) \Rightarrow \psi}{\theta, (\theta \rightarrow \varphi), (\theta \rightarrow \psi) \Rightarrow \varphi \wedge \psi}}{(\theta \rightarrow \varphi), (\theta \rightarrow \psi) \Rightarrow (\theta \rightarrow (\varphi \wedge \psi))}$$

Therefore,  $LL \vdash (\theta \rightarrow \varphi), (\theta \rightarrow \psi) \Rightarrow (\theta \rightarrow (\varphi \wedge \psi))$

(7) By using (4) and Rules  $\vee$ -left,  $\rightarrow$ -right,

$$\frac{\frac{\varphi, (\varphi \rightarrow \theta), (\psi \rightarrow \theta) \Rightarrow \theta \quad \psi, (\varphi \rightarrow \theta), (\psi \rightarrow \theta) \Rightarrow \theta}{(\varphi \vee \psi), (\varphi \rightarrow \theta), (\psi \rightarrow \theta) \Rightarrow \theta}}{(\varphi \rightarrow \theta), (\psi \rightarrow \theta) \Rightarrow ((\varphi \vee \psi) \rightarrow \theta)}$$

(8) By  $\neg$ -left and (1),  $LL \vdash \varphi \wedge \neg\varphi \Rightarrow$ .

(9) By  $\neg$ -left and then  $\neg$ -right. We also use the  $\Box$ -closedness of  $\neg\varphi$  and  $\bar{\Gamma}$ .

(10) Since  $\neg\varphi$  is  $\Box$ -closed, by  $\neg$ -right,

$$\frac{\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}}{\varphi \Rightarrow \neg\neg\varphi}$$

Therefore,

$$LL \vdash \varphi \Rightarrow \neg\neg\varphi.$$

By the  $\Box$ -closedness of  $\Box\varphi$ ,  $\vdash \neg\neg\Box\varphi \Rightarrow \Box\varphi$ . Therefore,

$$LL \vdash \Box\varphi \Leftrightarrow \neg\neg\Box\varphi.$$

(11) Since  $\neg(\varphi \vee \psi)$  is  $\Box$ -closed,

$$\frac{\frac{\frac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee \psi}}{\neg(\varphi \vee \psi), \varphi \Rightarrow}}{\neg(\varphi \vee \psi) \Rightarrow \neg\varphi}$$

Therefore,  $LL \vdash \neg(\varphi \vee \psi) \Rightarrow \neg\varphi$ . Similarly,  $LL \vdash \neg(\varphi \vee \psi) \Rightarrow \neg\psi$ . Hence,  $LL \vdash \neg(\varphi \vee \psi) \Rightarrow (\neg\varphi \wedge \neg\psi)$ . The converse is proved, by using the fact that  $\neg\varphi \wedge \neg\psi$  is  $\Box$ -closed, i.e.

$$\frac{\frac{\vdots}{\neg\varphi \wedge \neg\psi, \varphi \Rightarrow} \quad \frac{\vdots}{\neg\varphi \wedge \neg\psi, \psi \Rightarrow}}{\frac{\neg\varphi \wedge \neg\psi, \varphi \vee \psi \Rightarrow}{\neg\varphi \wedge \neg\psi \Rightarrow \neg(\varphi \vee \psi)}}$$

$\therefore LL \vdash \neg(\varphi \vee \psi) \Leftrightarrow (\neg\varphi \wedge \neg\psi)$ .

(12) Similarly to the first part of (13),

$$LL \vdash \neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi).$$

(13)

$$\frac{\frac{\varphi \Rightarrow \varphi}{\Rightarrow (\varphi \rightarrow \varphi)} \quad \varphi \Rightarrow \varphi}{((\varphi \rightarrow \varphi) \rightarrow \varphi) \Rightarrow \varphi}$$

$\therefore LL \vdash \Box\varphi \Rightarrow \varphi$ .

(14)

$$\frac{\frac{\vdots}{\Box\neg\varphi \Rightarrow \neg\varphi} \quad \frac{\vdots}{\neg\varphi, \varphi \Rightarrow}}{\frac{\Box\neg\varphi, \varphi \Rightarrow}{\varphi \Rightarrow \neg\Box\neg\varphi}}$$

$\therefore LL \vdash \varphi \Rightarrow \Diamond\varphi$ .

(15) If-part is obvious by (3). The converse follows from the fact:

$$\frac{\frac{\overline{\Gamma} \Rightarrow \overline{\Delta}, \varphi}{(\varphi \rightarrow \varphi), \overline{\Gamma} \Rightarrow \overline{\Delta}, \varphi}}{\overline{\Gamma} \Rightarrow \overline{\Delta}, ((\varphi \rightarrow \varphi) \rightarrow \varphi)}$$

(16) If  $\varphi$  is  $\Box$ -closed, then, by (17),  $LL \vdash \varphi \Rightarrow \Box\varphi$ . The converse is (15).

(17) Since  $\Box\varphi$  is  $\Box$ -closed,  $LL \vdash \Box\Box\varphi \Leftrightarrow \Box\varphi$ , by (18).

(18)  $LL \vdash \varphi, \overline{\Gamma} \Rightarrow \overline{\Delta}$  if and only if  $LL \vdash \Diamond\varphi, \overline{\Gamma} \Rightarrow \overline{\Delta}$

*Proof.* If-part is obvious by (16). The converse follows from:

$$\frac{\frac{\frac{\varphi, \bar{\Gamma} \Rightarrow \bar{\Delta}}{\bar{\Gamma} \Rightarrow \bar{\Delta}, \neg\varphi}}{\bar{\Gamma} \Rightarrow \bar{\Delta}, \Box\neg\varphi}}{\neg\Box\neg\varphi, \bar{\Gamma} \Rightarrow \bar{\Delta}}$$

□

$$(19) \quad \text{LL} \vdash \Box\varphi \wedge \exists x\psi(x) \Leftrightarrow \exists x(\Box\varphi \wedge \psi(x)) ; \quad \text{LL} \vdash \varphi \wedge \exists x\Box\psi(x) \Leftrightarrow \exists x(\varphi \wedge \Box\psi(x))$$

*Proof.*

$$\frac{\vdots}{\frac{\frac{\Box\varphi, \psi(a) \Rightarrow \Box\varphi \wedge \psi(a)}{\Box\varphi, \psi(a) \Rightarrow \exists x(\Box\varphi \wedge \psi(x))}}{\Box\varphi, \exists x\psi(x) \Rightarrow \exists x(\Box\varphi \wedge \psi(x))}}{\Box\varphi \wedge \exists x\psi(x) \Rightarrow \exists x(\Box\varphi \wedge \psi(x))}$$

$$\frac{\frac{\Box\varphi \Rightarrow \Box\varphi}{\Box\varphi \wedge \psi(a) \Rightarrow \Box\varphi} \quad \vdots}{\frac{\Box\varphi \wedge \psi(a) \Rightarrow \Box\varphi \wedge \exists x\psi(x)}{\exists x(\Box\varphi \wedge \psi(x)) \Rightarrow \Box\varphi \wedge \exists x\psi(x)}}$$

Similarly,  $\text{LL} \vdash (\varphi \wedge \exists x\Box\psi(x)) \Leftrightarrow \exists x(\varphi \wedge \Box\psi(x))$ .

□

$$(20) \quad \text{LL} \vdash \forall x\Box\varphi(x) \Leftrightarrow \Box\forall x\varphi(x)$$

*Proof.*  $(\Rightarrow)$  is obvious. Proof of  $(\Leftarrow)$  is:

$$\frac{\vdots}{\frac{\Box\forall x\varphi(x) \Rightarrow \Box\varphi(a)}{\Box\forall x\varphi(x) \Rightarrow \forall x\Box\varphi(x)}}$$

□

$$(21) \quad \text{LL} \vdash (\Box\varphi \rightarrow \psi) \Leftrightarrow (\Box\varphi \rightarrow \Box\psi) \Leftrightarrow (\neg\Box\varphi \vee \Box\psi)$$

*Proof.* Since the first  $(\Leftrightarrow)$  is obvious from (5), we prove only the second  $(\Leftrightarrow)$ .

$$\frac{\frac{\frac{\vdots}{(\Box\varphi \rightarrow \Box\psi), \Box\varphi \Rightarrow \Box\psi}}{(\Box\varphi \rightarrow \Box\psi) \Rightarrow \Box\psi, \neg\Box\varphi}}{(\Box\varphi \rightarrow \Box\psi) \Rightarrow \neg\Box\varphi \vee \Box\psi}$$

$$\frac{\frac{\Box\psi \Rightarrow \Box\psi}{\Box\varphi, \Box\psi \Rightarrow \Box\psi} \quad \frac{\vdots}{\neg\Box\varphi, \Box\varphi \Rightarrow \Box\psi}}{\frac{\neg\Box\varphi \vee \Box\psi, \Box\varphi \Rightarrow \Box\psi}{\neg\Box\varphi \vee \Box\psi \Rightarrow (\Box\varphi \rightarrow \Box\psi)}}$$

□

$$(22) \quad \text{LL} \vdash \Rightarrow \Box\varphi \vee \neg\Box\varphi$$

*Proof.* Obvious.

□

$$(23) \quad \text{LL} \vdash ((\varphi \wedge \Box\xi) \rightarrow \psi) \Rightarrow ((\neg\psi \wedge \Box\xi) \rightarrow \neg\varphi)$$

*Proof.*

$$\frac{\frac{\frac{\vdots}{\varphi, \Box\xi \Rightarrow \varphi \wedge \Box\xi}}{(\varphi \wedge \Box\xi \rightarrow \psi), \neg\psi, \varphi, \Box\xi \Rightarrow} \quad \frac{\frac{\vdots}{\psi, \neg\psi \Rightarrow}}{(\varphi \wedge \Box\xi \rightarrow \psi), \neg\psi \wedge \Box\xi \Rightarrow \neg\varphi}}{(\varphi \wedge \Box\xi \rightarrow \psi) \Rightarrow (\neg\psi \wedge \Box\xi \rightarrow \neg\varphi)}$$

□

$$(24) \quad \text{Since } \neg\varphi \text{ is } \Box\text{-closed,}$$

$$\text{LL} \vdash \neg\varphi \Leftrightarrow \Box\neg\varphi.$$

$$(25) \quad \text{LL} \vdash (\varphi \rightarrow \Box\psi) \Rightarrow (\Diamond\varphi \rightarrow \Box\psi)$$

*Proof.* By (7).

□

$$(26) \quad \text{LL} \vdash \Diamond(\Box\varphi \wedge \psi) \Rightarrow \Box\varphi \wedge \Diamond\psi$$

*Proof.* By (4) and (7). □

$$(27) \quad \text{LL} \vdash \exists x \Diamond \varphi(x) \Leftrightarrow \Diamond \exists x \varphi(x)$$

*Proof.* Similar to (20). □

□

#### 5.4.4 Interpretation of LL

Let  $\mathcal{L}$  be a complete lattice and  $M = \langle \mathcal{L}, D, I \rangle$  be a model of LL, where  $D$  is a domain of variables, and  $I$  is an interpretation of predicate symbols

$$I(p^n) : D^n \rightarrow \mathcal{L}.$$

$\rightarrow$  and  $\neg$  are interpreted as

$$\begin{aligned} (\varphi \rightarrow \psi)[M, v] &= (\varphi[M, v] \rightarrow \psi[M, v]) = \begin{cases} 1, & \text{if } \varphi[M, v] \leq \psi[M, v] \\ 0, & \text{otherwise} \end{cases} \\ (\neg \varphi)[M, v] &= \neg \varphi[M, v] = \begin{cases} 1, & \text{if } \varphi[M, v] = 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

That is,

$$\begin{aligned} \llbracket \varphi \rightarrow \psi \rrbracket &= (\llbracket \varphi \rrbracket \rightarrow \llbracket \psi \rrbracket) = \begin{cases} 1, & \llbracket \varphi \rrbracket \leq \llbracket \psi \rrbracket \\ 0, & \text{otherwise,} \end{cases} \\ \llbracket \neg \varphi \rrbracket &= \neg \llbracket \varphi \rrbracket = \begin{cases} 1, & \llbracket \varphi \rrbracket = 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

**THEOREM 5.4.3** (Soundness). *If a sequent  $\Gamma \Rightarrow \Delta$  is LL-provable, then it is valid in every lattice valued model.*

$$\text{i.e.} \quad \text{If } \vdash \Gamma \Rightarrow \Delta \text{ then } \models \Gamma \Rightarrow \Delta$$

*Proof.* Let an arbitrary lattice valued model  $M = \langle \mathcal{L}, D, I \rangle$  and D-assignment  $v$  be fixed. Axiom  $\varphi \Rightarrow \varphi$  is valid, since  $\llbracket \varphi \rrbracket \leq \llbracket \varphi \rrbracket$ . So it suffices to show that, for each inference rule

$$\frac{P_1, \dots, P_m}{Q}$$

if  $M \models P_i$  for  $i = 1, \dots, m$ , then  $M \models Q$ .

We prove the case of Cut, as an example. Since the truth value  $\llbracket \varphi \rrbracket$  of  $\square$ -closed formula is either 1 or 0, it suffices to show for the cases

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Lambda}, \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi \Rightarrow \Lambda}{\Gamma \Rightarrow \Delta, \Lambda}, \quad \frac{\Gamma \Rightarrow \Delta, \bar{\varphi} \quad \bar{\varphi}, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}.$$

Let  $\Gamma \Rightarrow \varphi$  and  $\varphi, \Pi \Rightarrow \Lambda$  are valid. If  $\Gamma = \{\varphi_1, \dots, \varphi_m\}$ ,  $\Pi = \{\psi_1, \dots, \psi_n\}$ ,  $\Lambda = \{\xi_1, \dots, \xi_l\}$ ,

$$\llbracket \varphi_1 \wedge \dots \wedge \varphi_m \rrbracket \leq \llbracket \varphi \rrbracket,$$

and

$$\llbracket \varphi \rrbracket \wedge \llbracket \psi_1 \wedge \dots \wedge \psi_n \rrbracket \leq \llbracket \xi_1 \vee \dots \vee \xi_l \rrbracket.$$

Then

$$\llbracket \varphi_1 \wedge \dots \wedge \varphi_m \rrbracket \wedge \llbracket \psi_1 \wedge \dots \wedge \psi_n \rrbracket \leq \llbracket \xi_1 \vee \dots \vee \xi_l \rrbracket.$$

That is, the lower sequent is also valid.

Other cases are proved similarly. □

**COROLLARY 5.4.4.** *If a generalized sequent is LL-provable, then it is valid in every lattice valued model.*

*Proof.* It follows from the fact that a generalized sequent  $\Phi \Rightarrow \Psi$  is LL-provable, if there exist finite subsequences  $\Gamma \subset \Phi$  and  $\Delta \subset \Psi$  such that

$$\text{LL} \vdash \Gamma \Rightarrow \Delta.$$

□

Now we review the proof of the strong completeness of LL by M.Takano.

**THEOREM 5.4.5** (Strong completeness, Takano [10]). *If a generalized sequent is valid in lattice valued model, then it is LL-provable:*

$$\text{i.e.} \quad \text{If } \models \Phi \Rightarrow \Psi \text{ then } \vdash \Phi \Rightarrow \Psi$$

*Proof.* To prove the theorem, a generalized sequent  $\Phi \Rightarrow \Psi$  is supposed to be LL-unprovable, and construct a lattice valued model  $M = \langle \mathcal{L}, D, I \rangle$  and  $D$ -assignment  $v$  which falsifies

$$\bigwedge \{ \llbracket \varphi \rrbracket \mid \varphi \in \Phi \} \leq \bigvee \{ \llbracket \psi \rrbracket \mid \psi \in \Psi \}.$$

We assume without loss of generality, that  $p$  and  $q$  are mutually distinct proposition symbols (namely, 0-ary predicate symbols) which do not occur in  $\Phi \cup \Psi$  and that there are infinitely many free variables which do not occur in  $\Phi \cup \Psi$ .

A formula of the form  $(\varphi \rightarrow \psi)$  is called an **implication formula**. Let  $\theta_0, \theta_1, \theta_2, \dots$  be an enumeration of all implication formulas. The sets  $\Phi_k$  and  $\Psi_k$  ( $k = 0, 1, 2, \dots$ ) of implication formulas are defined recursively as follows, so as to have infinitely many free variables which do not occur in  $\Phi_k \cup \Psi_k$ .

**Basis.** Define  $\Phi_0$  and  $\Psi_0$  as follows:

$$\Phi_0 = \{(p \rightarrow \varphi) \mid \varphi \in \Phi\} \cup \{(\psi \rightarrow q) \mid \psi \in \Psi\} \quad \text{and} \quad \Psi_0 = \{(p \rightarrow q)\}.$$

**Induction Step.** Suppose that  $\Phi_k$  and  $\Psi_k$  have been defined. Let  $a$  be any free variable which does not occur in  $\Phi_k \cup \Psi_k \cup \{\theta_k\}$ .

**Case 1:**  $\Phi_k \Rightarrow \Psi_k \cup \{\theta_k\}$  is LL-provable. Define

$$\Phi_{k+1} = \Phi_k \cup \{\theta_k\}, \quad \Psi_{k+1} = \Psi_k.$$

**Case 2 :**  $\Phi_k \Rightarrow \Psi_k \cup \{\theta_k\}$  is LL-unprovable.

(1) If  $\theta_k$  is of the form  $(\psi \rightarrow \forall x \varphi(x))$ , define

$$\Phi_{k+1} = \Phi_k \quad \Psi_{k+1} = \Psi_k \cup \{\theta_k, (\psi \rightarrow \varphi(a))\}$$

(2) If  $\theta_k$  is of the form  $(\exists x \varphi(x) \rightarrow \psi)$ , define

$$\Phi_{k+1} = \Phi_k \quad \Psi_{k+1} = \Psi_k \cup \{\theta_k, (\varphi(a) \rightarrow \psi)\}.$$

(3) Otherwise, define

$$\Phi_{k+1} = \Phi_k \quad \Psi_{k+1} = \Psi_k \cup \{\theta_k\}.$$

Having defined all  $\Phi_k$ 's and  $\Psi_k$ 's, let

$$\Phi_\infty = \bigcup \{\Phi_k \mid k = 0, 1, \dots\} \quad \text{and} \quad \Psi_\infty = \bigcup \{\Psi_k \mid k = 0, 1, \dots\}.$$

**PROPOSITION 5.4.6.** *The following properties hold for the set  $\Phi$  of implication formulas.*

- (1) *Suppose that  $\varphi_1, \dots, \varphi_m, \psi_1, \dots, \psi_n$  are implication formulas, and*

$$\text{LL} \vdash \varphi_1, \dots, \varphi_m \Rightarrow \psi_1, \dots, \psi_n.$$

*If  $\varphi_1, \dots, \varphi_m \in \Phi_\infty$ , then either  $\psi_1 \in \Phi_\infty$  or  $\dots$  or  $\psi_n \in \Phi_\infty$ .*

- (2) *If  $(\psi \rightarrow \varphi(a)) \in \Phi_\infty$  for every  $a \in FV$ , then  $(\psi \rightarrow \forall x \varphi(x)) \in \Phi_\infty$*

- (3) *If  $(\varphi(a) \rightarrow \psi) \in \Phi_\infty$  for every  $a \in FV$ , then  $(\exists x \varphi(x) \rightarrow \psi) \in \Phi_\infty$*

- (4)  *$(p \rightarrow \varphi) \in \Phi_\infty$  for every  $\varphi \in \Phi$ , while  $(p \rightarrow q) \notin \Phi$ .*

*Proof.* (1) If  $\varphi_1, \dots, \varphi_m \in \Phi_\infty$  but  $\psi_1, \dots, \psi_n \notin \Phi_\infty$ , then

$$\text{LL} \not\vdash \varphi_1, \dots, \varphi_m \Rightarrow \psi_1, \dots, \psi_n,$$

which contradicts to the assumption.

- (2) Suppose that  $(\psi \rightarrow \forall x \varphi(x)) = \theta_k$ , and  $\theta_k \notin \Phi_\infty$ . If  $\text{LL} \vdash \Phi_k \Rightarrow \Psi_k \cup \{\theta_k\}$ , then  $\theta_k \in \Phi_{k+1}$  contradicting  $\theta_k \notin \Phi_\infty$ ; hence  $\text{LL} \not\vdash \Phi_k \Rightarrow \Psi_k \cup \{\theta_k\}$ . It follows that for some  $a \in FV$ ,  $(\psi \rightarrow \varphi(a)) \in \Psi_{k+1}$ , and so  $(\psi \rightarrow \varphi(a)) \in \Phi_\infty$ .

- (3) Similar to (2).

- (4)  $(p \rightarrow \varphi) \in \Phi_0$  for every  $\varphi \in \Phi$ , and  $(\psi \rightarrow q) \in \Phi_0$  for every  $\psi \in \Psi$ . On the other hand,  $(p \rightarrow q) \notin \Phi_\infty$ , since  $(p \rightarrow q) \in \Psi_0$ .

□

**PROPOSITION 5.4.7.** *Every implication formula belongs to exactly one of  $\Phi_\infty$  and  $\Psi_\infty$ , and the generalized sequent  $\Phi_\infty \Rightarrow \Psi_\infty$  is LL-unprovable.*

*Proof.* Since  $\theta_k \in \Phi_{k+1} \cup \Psi_{k+1}$ , every implication formula belongs to at least one of  $\Phi_\infty$  and  $\Psi_\infty$ . So it suffices to show that  $\Phi_k \Rightarrow \Psi_k$  is LL-unprovable for every natural number  $k$ . We prove this by induction on  $k$ .

**Basis.** Suppose that  $\Phi_0 \Rightarrow \Psi_0$  were LL-provable. then the sequent

$$(p \rightarrow \varphi_1), \dots, (p \rightarrow \varphi_m), (\psi_1 \rightarrow q), \dots, (\psi_n \rightarrow q) \Rightarrow (p \rightarrow q)$$

is LL-provable for some  $\varphi_1, \dots, \varphi_m \in \Phi$  and some  $\psi_1, \dots, \psi_n \in \Psi$ . Replacing  $p$  and  $q$  by  $\varphi_1 \wedge \dots \wedge \varphi_m$  ( $p \rightarrow p$ , if  $m=0$ ) and  $\psi_1 \wedge \dots \wedge \psi_n$  ( $q \rightarrow q$ , if  $n=0$ ), respectively, the sequent

$$\varphi'_1, \dots, \varphi'_m, \psi'_1, \dots, \psi'_n \Rightarrow ((\varphi_1 \wedge \dots \wedge \varphi_m) \rightarrow (\psi_1 \wedge \dots \wedge \psi_n))$$

is LL-provable, where  $\varphi'_i$  and  $\psi'_j$  denote  $((\varphi_1 \wedge \dots \wedge \varphi_m) \rightarrow \varphi_i)$  and  $(\psi_j \rightarrow (\psi_1 \vee \dots \vee \psi_n))$ , respectively ( $i = 1, \dots, m$ ;  $j = 1, \dots, n$ ). Then by Theorem 5.4.2(6),(7),

$$\text{LL} \vdash \varphi_1, \dots, \varphi_m \Rightarrow \psi_1, \dots, \psi_n$$

which contradicts the assumption that  $\Phi \Rightarrow \Psi$  is LL-unprovable. Hence  $\Phi_0 \Rightarrow \Psi_0$  is LL-unprovable.

**Induction Step** Suppose that  $\Phi_k \Rightarrow \Psi_k$  is LL-unprovable. We will show that  $\Phi_{k+1} \Rightarrow \Psi_{k+1}$  is LL-unprovable, along the definition of  $\Phi_{k+1}$  and  $\Psi_{k+1}$ .

**Case 1 :**  $\Phi_k \Rightarrow \Psi_k \cup \{\theta_k\}$  is LL-provable. If  $\Phi_{k+1}$  and  $\Psi_{k+1}$  were LL-provable, then  $\Phi_k$  and  $\Psi_k$  is also LL-provable by Cut, contradicting the assumption. Hence,  $\Phi_{k+1}$  and  $\Psi_{k+1}$  were LL-unprovable.

**Case 2 :**  $\Phi_k \Rightarrow \Psi_k \cup \{\theta_k\}$  is LL-unprovable.

If  $\theta_k$  is of the form  $\psi \supset \forall x \varphi(x)$ , then

$$\Phi_{k+1} = \Phi_k \quad \Psi_{k+1} = \Psi_k \cup \{\theta_k, (\psi \rightarrow \varphi(a))\}$$

If  $\theta_k$  is of the form  $\exists x \varphi(x) \supset \psi$ , then

$$\Phi_{k+1} = \Phi_k \quad \Psi_{k+1} = \Psi_k \cup \{\theta_k, (\varphi(a) \rightarrow \psi)\}.$$

Since  $\Phi_{k+1}$  and  $\Psi_{k+1}$  are sets of closed formulas,  $\Phi_{k+1}$  and  $\Psi_{k+1}$  were LL-unprovable by the inference rules  $\forall$ -right,  $\exists$ -right and  $\rightarrow$ -right.

**Otherwise :**  $\not\vdash \Phi_{k+1} \Rightarrow \Psi_{k+1}$  is obvious.

□

Let

$$\varphi \equiv \psi \stackrel{\text{def}}{\iff} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \in \Phi_\infty$$

$\equiv$  is an equivalence relation by Proposition 5.4.6. Let  $|\varphi|$  be the equivalence class under the equivalence relation to which  $\varphi$  belongs :

$$|\varphi| \stackrel{\text{def}}{=} \{\xi : \text{formula} \mid \xi \equiv \varphi\}.$$

Set  $P$  of all the equivalence classes under  $\equiv$  :

$$P \stackrel{\text{def}}{=} \{|\varphi| \mid \varphi : \text{formula}\}$$

is an ordered set under  $\leq$  such that

$$|\varphi| \leq |\psi| \iff (\varphi \rightarrow \psi) \in \Phi_\infty \quad \text{for formulas } \varphi, \psi.$$

**PROPOSITION 5.4.8.**  $\langle P, \leq \rangle$  has the following properties.

- (1)  $P$  has both the greatest element 1 and the least element 0, and  $1 \neq 0$ .
- (2)  $|\varphi \wedge \psi| = |\varphi| \wedge |\psi|$ .
- (3)  $|\varphi \vee \psi| = |\varphi| \vee |\psi|$ .
- (4)  $|\varphi \rightarrow \psi| = (|\varphi| \rightarrow |\psi|)$ .
- (5)  $|\neg \varphi| = \begin{cases} 1, & |\varphi| = 1, \\ 0, & \text{otherwise.} \end{cases}$
- (6)  $|\forall x \varphi(x)| = \bigwedge \{|\varphi(a)| \mid a \in FV\}$ .
- (7)  $|\exists x \varphi(x)| = \bigvee \{|\varphi(a)| \mid a \in FV\}$ .

*Proof.* (1) Since  $\text{LL} \vdash \Rightarrow (\text{p} \rightarrow \text{p})$ ,

$$\text{LL} \vdash \theta \Rightarrow (\text{p} \rightarrow \text{p})$$

for every  $\theta$ . Hence,  $|p \rightarrow p|$  is the greatest. Similarly

$$\text{LL} \vdash \Rightarrow (\neg(\text{p} \rightarrow \text{p}) \rightarrow \theta)$$

for every  $\theta$ . Hence,  $|\neg(p \rightarrow p)|$  is the least.

$$|p \rightarrow p| = 1 \quad \text{and} \quad |\neg(p \rightarrow p)| = 0.$$

By Proposition 5.4.6 (4),  $(p \rightarrow q) \notin \Phi_\infty$ . Hence  $|p| \neq |q|$ . Therefore,  $1 \neq 0$ .

- (2) By Theorem 5.4.2 (6) and (8),  $|\varphi \wedge \psi| \leq |\varphi| \wedge |\psi|$ , and if  $|\xi| \leq |\varphi|$  and  $|\xi| \leq |\psi|$  then  $|\xi| \leq |\varphi \wedge \psi|$ .
- (3) Similar to (3) by Theorem 5.4.2 (7) and (9).
- (4) Since  $\varphi \rightarrow \psi$  is  $\Box$ -closed,

$$\frac{\frac{(\varphi \rightarrow \psi) \Rightarrow (\varphi \rightarrow \psi)}{(\varphi \rightarrow \psi) \Rightarrow (\varphi \rightarrow \psi), \theta}}{\Rightarrow (\varphi \rightarrow \psi), ((\varphi \rightarrow \psi) \rightarrow \theta)}$$

is a proof for every  $\theta$ . By Proposition 5.4.6 (1),

$$\text{either } (\varphi \rightarrow \psi) \in \Phi_\infty, \text{ or } ((\varphi \rightarrow \psi) \rightarrow \theta) \in \Phi_\infty$$

Hence, for every  $\theta$ ,

$$\text{either } (\theta \rightarrow (\varphi \rightarrow \psi)) \in \Phi_\infty, \text{ or } ((\varphi \rightarrow \psi) \rightarrow \theta) \in \Phi_\infty$$

It follows that

$$\text{either } |\varphi \rightarrow \psi| = 1 \text{ or } |\varphi \rightarrow \psi| = 0.$$

Since  $\text{LL} \vdash \theta \Rightarrow (p \rightarrow p)$  for every  $\theta$ ,  $p \rightarrow p$  is in  $\Phi_\infty$ . and Theorem 5.4.2 (10) and (11).

- (5) If  $|\varphi| = 0$ , then  $|\varphi| \leq |\neg\varphi|$  in particular, i.e.  $(\varphi \rightarrow \neg\varphi) \in \Phi_\infty$ .

For every  $\psi$ , we have

$$\frac{\frac{\frac{\varphi \Rightarrow \varphi \quad \neg\varphi \Rightarrow \neg\varphi}{\varphi, (\varphi \rightarrow \neg\varphi) \Rightarrow \neg\varphi}}{(\varphi \rightarrow \neg\varphi) \Rightarrow \neg\varphi}}{(\varphi \rightarrow \neg\varphi) \Rightarrow (\psi \rightarrow \neg\varphi),}$$

$$\therefore \quad \text{LL} \vdash (\varphi \rightarrow \neg\varphi) \Rightarrow (\psi \rightarrow \neg\varphi).$$

Hence, If  $|\varphi| = 0$ , then  $|\psi| \leq |\neg\varphi|$  for every  $\psi$ . That is,

$$\text{if } |\varphi| = 0, \text{ then } |\neg\varphi| = 1.$$

If  $|\varphi| \neq 0$ , then  $|\varphi| \not\leq |\psi|$  for some  $\psi$ . That is,  $(\varphi \rightarrow \psi) \notin \Phi_\infty$  for some  $\psi$ . But we have  $\text{LL} \vdash \Rightarrow (\varphi \rightarrow \psi), (\neg\varphi \rightarrow \theta)$ , for every  $\psi$  and  $\theta$ . Because,

$$\frac{\varphi, \neg\varphi \Rightarrow}{\varphi, \neg\varphi \Rightarrow \psi, \theta} \frac{}{\Rightarrow (\varphi \rightarrow \psi), (\neg\varphi \rightarrow \theta)}.$$

Since  $(\varphi \rightarrow \psi) \notin \Phi_\infty$ ,  $(\neg\varphi \rightarrow \theta) \in \Phi_\infty$  for every  $\theta$ . It follows that  $|\neg\varphi| = 0$

(6) Since  $\text{LL} \vdash \forall x\varphi(x) \Rightarrow \varphi(a)$ , hence

$$|\forall x\varphi(x)| \leq |\varphi(a)| \text{ for every } a \in FV.$$

On the other hand, by Proposition 5.4.6 (3),

$$|\psi| \leq |\varphi(a)| \text{ for every } a \in FV \text{ implies } |\psi| \leq |\forall x\varphi(x)|.$$

$$\therefore \quad |\forall x\varphi(x)| = \bigwedge \{|\varphi(a)| \mid a \in FV\}.$$

(7) Similar to (6).

□

**LEMMA 5.4.9.** [McNeille's theorem ([5], [2])] Any poset  $P$  can be embedded in a complete lattice  $\mathcal{L}$ , so that order is preserved, together with the suprema and infima existing in  $P$ , where for each  $X \subset P$ ,

$X^*$  denotes the set of upper bounds in  $P$ , i.e.  $X^* = \{x \in P \mid \forall t \in X (t \leq x)\}$ ,

$X^\dagger$  denotes the set of lower bounds in  $P$ , i.e.  $X^\dagger = \{x \in P \mid \forall t \in X (x \leq t)\}$

and

$\mathcal{L}$  is defined to be the set of all  $X \subset P$  such that  $X = (X^*)^\dagger$ :

$$\mathcal{L} \stackrel{\text{def}}{=} \{X \subset P \mid X = (X^*)^\dagger\}.$$

Then

$$\bigwedge \{X_i\}_i = \bigcap_i X_i \quad \text{and} \quad \bigvee \{X_i\}_i = ((\bigcup_i X_i)^*)^\dagger \quad \text{for } \{X_i\}_i \subset \mathcal{L}.$$

The embedding  $i : P \rightarrow \mathcal{L}$  is defined by  $i(a) = \{t \in P \mid t \leq a\}$  for  $a \in P$ .

Now, consider the lattice valued model  $M = \langle \mathcal{L}, D, f \rangle$ , where  $\mathcal{L}$  is the complete lattice in which  $P$  is embedded,  $D$  is  $FV$ , and for every predicate symbol  $R$ ,  $f(R)$  is the mapping from  $D^n$  to  $\mathcal{L}$  such that

$$f(R)(a_1, \dots, a_n) = i(|R(a_1, \dots, a_n)|) \quad \text{for every } a_1, \dots, a_n \in D.$$

**PROPOSITION 5.4.10.** *For every formula  $\varphi$  and every  $D$ -assignment  $v : FV \rightarrow D$ ,*

$$\varphi[M, v] = i(|\varphi^v|),$$

where  $\varphi^v$  designates the formula obtained from  $\varphi$  by substituting every occurrence of each free variable  $a$  by free variable  $v(a)$ .

*Proof.* The proof is by induction on the construction of  $\varphi$ .

Case 1:  $\varphi$  is  $R(a_1, \dots, a_n)$ , where  $R$  is an  $n$ -ary predicate symbol. Then  $\varphi^v$  is  $R(v(a_1), \dots, v(a_n))$ . So

$$\varphi[M, v] = f(R)((v(a_1), \dots, v(a_n))) = i(|\varphi^v|).$$

Case 2:  $\varphi$  is  $\psi \wedge \theta$ . By the hypothesis of induction,  $\psi[M, v] = i(|\psi^v|)$  and  $\theta[M, v] = i(|\theta^v|)$ . On the other hand, by Proposition 5.4.8(2), and by the fact that  $i$  preserves the infimums,  $i(|\varphi^v|) = i(|\psi^v|) \wedge i(|\theta^v|)$ . So

$$\varphi[M, v] = \psi[M, v] \wedge \theta[M, v] = i(|\psi^v|) \wedge i(|\theta^v|) = i(|\varphi^v|).$$

Case 3-5:  $\varphi$  is  $(\psi \vee \theta)$ ,  $(\psi \rightarrow \theta)$  or  $\neg\psi$ . Similar to Case 2.

Case 6:  $\varphi$  is  $\forall x \psi(x)$ . Then  $\varphi^v$  is  $\forall x \psi^v(x)$ . By hypothesis of induction,

$$\psi(a)[M.v(d/a)] = i(|\psi^v(d)|) \quad \text{for every } d \in D.$$

On the other hand, by Proposition 5.4.8(6) ,

$$|\varphi^v| = \bigwedge \{|\psi^v(d)| \mid d \in D\} \quad \therefore \quad i(|\varphi^v|) = \bigwedge \{i(|\psi^v(d)|) \mid d \in D\},$$

since  $i$  preserves the infimums. Hence,

$$\varphi[M, v] = \bigwedge \{ \psi(a)[M.v(d/a)] \mid d \in D \} = \bigwedge \{ i(|\psi^v(d)|) \mid d \in D \} = i(|\varphi^v|)$$

Case 7:  $\varphi$  is  $\exists x\psi(x)$ . Similar to Case 6.  $\square$

Let  $w$  be the  $D$ -assignment which is the identity mapping on  $D$ . It follows from the previous proposition that  $\varphi[M, w] = i(|\varphi|)$  for every  $\varphi$ .

**PROPOSITION 5.4.11.**  $\bigwedge \{ \varphi[M, w] : \varphi \in \Phi \} \not\leq \bigvee \{ \Psi[M, w] : \psi \in \Psi \}$ .

*Proof.* It suffices to show that

$$|p| \in \bigwedge \{ \varphi[M, w] : \varphi \in \Phi \} \quad \text{but} \quad |p| \notin \bigvee \{ \Psi[M, w] : \psi \in \Psi \}.$$

If  $\varphi \in \Phi$ , then  $(p \rightarrow \varphi) \in \Phi_\infty$  by Proposition 5.4.6(4), so  $|p| \leq |\varphi|$  and so  $|p| \in i(|\varphi|) = \varphi[M, w]$ . Hence

$$|p| \in \bigcap \{ \varphi[M, w] : \varphi \in \Phi \} = \bigwedge \{ \varphi[M, w] : \varphi \in \Phi \}.$$

If  $\psi \in \Psi$ , in the meantime,  $(\psi \rightarrow q) \in \Phi_\infty$  by Proposition 5.4.6(4), so  $|\psi| \leq |q|$  and so  $|q| \in (i(|\psi|))^* = (\psi[M, w])^*$ . Hence  $|q| \in \bigcap \{ (\psi[M, w])^* \mid \psi \in \Psi \}$ . But, since  $(p \rightarrow q) \notin \Phi_\infty$  by Proposition 5.4.6(4) again, it is not the case that  $|p| \leq |q|$ . So

$$|p| \notin \left( \bigcap \{ (\psi[M, w])^* \mid \psi \in \Psi \} \right)^\dagger = \bigvee \{ \psi[M, w] \mid \psi \in \Psi \}.$$

$\square$

Hence,

$$\bigwedge \{ \varphi[M, v] \mid \varphi \in \Phi \} \not\leq \bigvee \{ \psi[M, v] \mid \psi \in \Psi \},$$

That is,  $\Phi \Rightarrow \Psi$  is not valid. this ends the proof of Theorem 5.4.5.

$\square$

## 5.5 Basic implication and globalization

Lattice valued logic and lattice valued set theory are formulated in Titani [13] by introducing **basic implication**  $\rightarrow$ , which represents lattice order on the truth value set  $\mathcal{L}$ :

$$(a \rightarrow b) = \begin{cases} 1 & a \leq b \\ 0 & a \not\leq b, \end{cases}$$

where 1 and 0 represents ‘true’ and ‘false’, respectively.

We assume that the truth values of the meta-logic are ‘true’ and ‘false’, that is, the sublattice  $\mathbf{2} = \{1, 0\}$ , which is a Boolean algebra, represents the truth value set of the meta-logic. Thus, the meta-logic is a classical logic.

Logical **operation** is represented as an algebraic operation on  $\mathcal{L}$ :

$$\mathcal{L} \times \cdots \times \mathcal{L} \rightarrow \mathcal{L},$$

whereas **relation** is represented by an algebraic relation,

$$\mathcal{L} \times \cdots \times \mathcal{L} \rightarrow \mathbf{2} (\subset \mathcal{L}).$$

Since  $\mathbf{2}$  is a sub-lattice of every complete lattice, the basic implication  $\rightarrow$  can be considered as an algebraic operation on the complete lattice  $\mathcal{L}$ . So we could introduce  $\rightarrow$  into logic as a logical operation.

Globalization  $\Box$

$$\Box a = \begin{cases} 1 & a = 1 \\ 0 & a \neq 1 \end{cases}$$

is a modal operator on the lattice, satisfying

- (1)  $\Box a \leq a$ ;
- (2)  $\Box \Box a = \Box a$ ;
- (3)  $\Box a \vee (\Box a)^\perp = 1$ ,  $\Box a \wedge (\Box a)^\perp = 0$ ; and
- (4)  $a \leq b \Rightarrow \Box a \leq \Box b$ .

The corresponding logical operation  $\Box$  is also called a **globalization**.

Thus, if  $\supset$  is an implication, then

$$(\varphi \rightarrow \psi) \stackrel{\text{def}}{\iff} \Box(\varphi \supset \psi).$$

Furthermore,  $\Diamond$  is defined by  $\Diamond \varphi \stackrel{\text{def}}{=} (\Box \varphi^\perp)^\perp$ .

### 5.5.1 Global classical logic and global intuitionistic logic

Global classical logic GLK and global intuitionistic logic GLJ are obtained from LK and LJ by introducing globalization.

Formula of the form  $\Box\varphi$  is said to be  **$\Box$ -closed**. sequence of  $\Box$ -closed formulas  $\{\Box\varphi_i\}_i$  is written as  $\Box(\{\varphi_i\}_i)$ .

### 5.5.2 Inference rules

**Structural rules :**

$$\text{Thinning :} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \quad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

where  $\Delta$  is  $\Box$ -closed for  
the intuitionistic logic

$$\text{Contraction :} \quad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$\text{Interchange :} \quad \frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda}$$

$$\text{Cut :} \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

**Logical rules:**

$$\supset : \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Lambda}{\varphi \supset \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda} \quad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$\wedge : \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$\begin{array}{l} \vee : \quad \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \\[10pt] \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \\[10pt] \neg : \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \varphi} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg \varphi, \Gamma \Rightarrow \Delta} \end{array}$$

where  $\Delta$  is  $\Box$ -closed for  
the intuitionistic logic

$$\begin{array}{l} \forall : \quad \frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)} \\[10pt] \text{where } t \text{ is any term} \qquad \text{where } a \text{ is a free variable which does} \\ \qquad \qquad \qquad \text{not occur in the lower sequent.} \\[10pt] \exists : \quad \frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x \varphi(x), \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)} \\[10pt] \text{where } a \text{ is a free variable which does} \qquad \text{where } t \text{ is any term} \\ \text{not occur in the lower sequent.} \end{array}$$

$$(\Box) : \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box \varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Box \Gamma \Rightarrow \Box \Delta, \varphi}{\Box \Gamma \Rightarrow \Box \Delta, \Box \varphi} \quad [\text{Globalization}].$$

Of course it can be formulated using the basic operation  $\rightarrow$  instead of  $\Box$ .

**DEFINITION 5.5.1.**

$\text{GLK} \vdash \Gamma \Rightarrow \Delta$  denote “a sequent  $\Gamma \Rightarrow \Delta$  is provable in GLK ”

$\text{GLJ} \vdash \Gamma \Rightarrow \Delta$  denote “a sequent  $\Gamma \Rightarrow \Delta$  is provable in GLJ ”

If GLK, GLJ are obvious, we write just

$$\vdash \Gamma \Rightarrow \Delta.$$

Obviously we have

**THEOREM 5.5.1.** (1) *If  $\text{LK} \vdash \Gamma \Rightarrow \Delta$ , then  $\text{GLK} \vdash \Gamma \Rightarrow \Delta$*

(2) *If  $\text{LJ} \vdash \Gamma \Rightarrow \Delta$ , then  $\text{GLJ} \vdash \Gamma \Rightarrow \Delta$*

(3) (*Soundness*) *If  $\vdash \Gamma \Rightarrow \Delta$  then  $\models \Gamma \Rightarrow \Delta$*

### 5.5.3 Lindenbaum algebra

Let  $F$  be the set of formulas, and for  $\varphi, \psi \in F$  let

$$\varphi \equiv \psi \stackrel{\text{def}}{\iff} \vdash \varphi \Leftrightarrow \psi.$$

$\equiv$  is an equivalence relation on the set  $F$  of formulas. The equivalence class of  $\varphi$  is denoted by  $|\varphi|$ , and let  $F/\equiv$  be the quotient space of  $F$  by  $\equiv$ .

$$F/\equiv \stackrel{\text{def}}{=} \{|\varphi| \subset F \mid \varphi \in F\}$$

The order relation  $\leq$  on  $F/\equiv$  is definable by

$$|\varphi| \leq |\psi| \stackrel{\text{def}}{\iff} \vdash \varphi \Rightarrow \psi.$$

**LEMMA 5.5.2.** *For  $\varphi, \psi \in F$ ,*

$$(1) \quad |\varphi| \wedge |\psi| = |\varphi \wedge \psi|,$$

$$(2) \quad |\varphi| \vee |\psi| = |\varphi \vee \psi|,$$

$$(3) \quad \neg|\varphi| = |\neg\varphi|,$$

$$(4) \quad |\forall x\varphi(x)| = \bigwedge\{|\varphi(t)| \mid t \in T\},$$

$$(5) \quad |\exists x\varphi(x)| = \bigvee\{|\varphi(t)| \mid t \in T\}, \text{ where } T \text{ is the set of all terms, i.e. individual free variables and constants.}$$

*Proof.* We prove only (1) and (4). Other equations are proved similarly.

(1) Since we have  $\vdash \varphi \wedge \psi \Rightarrow \varphi$  and  $\vdash \psi \wedge \psi \Rightarrow \psi$ ,

$$|\varphi \wedge \psi| \leq |\varphi| \quad \text{and} \quad |\varphi \wedge \psi| \leq |\psi|.$$

If  $|\xi| \leq |\varphi|$  and  $|\xi| \leq |\psi|$  for  $\xi \in F$ , then

$$\vdash \xi \Rightarrow \varphi \quad \text{and} \quad \vdash \xi \Rightarrow \psi, \quad \therefore \vdash \xi \Rightarrow \varphi \wedge \psi.$$

It follows that  $|\xi| \leq |\varphi \wedge \psi|$ . Therefore,

$$|\varphi| \wedge |\psi| = |\varphi \wedge \psi|.$$

(4) Since  $\vdash \forall x\varphi(x) \Rightarrow \varphi(t)$  for all term  $t \in T$ ,

$$|\forall x\varphi(x)| \leq |\varphi(t)| \quad \text{for all } t \in T.$$

Assume  $|\xi| \leq |\varphi(t)|$  for all term  $t \in T$ . That is,  $\vdash \xi \Rightarrow \varphi(t)$  for all  $t \in T$ . Choose a free variable  $a \in FV$  which does not occur in  $\xi \Rightarrow \forall x\varphi(x)$ . Then by inference rule  $\forall$ -right,

$$\vdash \xi \Rightarrow \forall x\varphi(x), \quad \text{i.e.} \quad |\xi| \leq |\forall x\varphi(x)|.$$

Therefore,

$$|\forall x\varphi(x)| = \bigwedge \{|\varphi(a)| \mid a \in FV\}.$$

□

Thus,  $\wedge$ ,  $\vee$  and  $\neg$  are defined on  $F/\equiv$ .

$F/\equiv$  is a Boolean algebra or Heyting algebra, according as GLK or GLJ, which is called a **Lindenbaum algebra**.

### 5.5.4 Completeness

**THEOREM 5.5.3** (Completeness of GLK). *If a sequent  $\Gamma \Rightarrow \Delta$  is valid in every Boolean valued model, then it is provable in GLK.*

*Proof.* Lindenbaum algebra  $F/\equiv$  of GLK is a Boolean algebra, which is extended to a complete Boolean algebra by Theorem 3.3.6,

Let  $B$  be the minimal extension (cf. Theorem 3.3.6, p.30) of the Boolean algebra  $F/\equiv$ , and the canonical isomorphism  $h^* : F/\equiv \rightarrow B$  preserves all infinite joins and meets, i.e.

$$\text{if } a = \bigvee_{t \in T}^{F/\equiv} a_t, \text{ then } h^*(a) = \bigvee_{t \in T}^B h^*(a_t), \quad (5.5.1)$$

$$\text{if } a = \bigwedge_{t \in T}^{F/\equiv} a_t, \text{ then } h^*(a) = \bigwedge_{t \in T}^B h^*(a_t). \quad (5.5.2)$$

For a formula  $\varphi \in F$ , let

$$\llbracket \varphi \rrbracket \stackrel{\text{def}}{=} h^*(|\varphi|) \in B.$$

if  $\text{GLK} \not\vdash \varphi$ , then  $|\varphi| \neq 1$ . therefore,  $\llbracket \varphi \rrbracket \neq 1$ , i.e.  $\langle B, D, I \rangle$  forms a model of GLK such that

$$\langle B, D, I \rangle \not\vdash \varphi,$$

where

$D$  is a domain of individual variables, which contains the individual constants,

$I$  is an interpretation  $I$  of predicate symbols

$$I(p^n) : D^n \rightarrow B,$$

and  $D$ -assignment

$$v : FV \rightarrow D.$$

□

**THEOREM 5.5.4** (Completeness of GLJ). *If a sequent  $\Gamma \Rightarrow \Delta$  is valid in every Heyting valued model, then it is provable in GLJ.*

*Proof.* Similar to the case of GLK, where we use Theorem 3.4.7 instead of Theorem 3.3.6. That is, order of truth values is defined by

$$\llbracket \varphi \rrbracket \leq \llbracket \psi \rrbracket \stackrel{\text{def}}{\iff} \text{GLJ} \vdash \varphi \Rightarrow \psi,$$

then the set of truth values is a Heyting algebra, and  $\supset$  corresponds to an implication on the Heyting algebra.

Heyting algebra is extended to a complete Heyting algebra by Theorem 3.4.7.

Let  $(F/\equiv)^*$  is the complete extension of  $F/\equiv$ .

$\langle (F/\equiv)^*, D, I \rangle$  forms a model of GLJ according to  $F/\equiv$ , where

$D$  is a domain of individual variables, which contains the individual constants,

$I$  is an interpretation  $I$  of predicate symbols

$$I(p^n) : D^n \rightarrow (F/\equiv)^*,$$

and  $D$ -assignment

$$v : FV \rightarrow D.$$

□

## 5.6 Predicate orthologic

**Ortholattice**( cf. Definition 3.5.1) is a lattice provided with ortho-complementation  $^\perp$  which satisfies the following conditions.

$$(C1) \quad a^{\perp\perp} = a,$$

$$(C2) \quad a \vee a^\perp = 1, \quad a \wedge a^\perp = 0,$$

$$(C3) \quad a \leq b \implies b^\perp \leq a^\perp.$$

The structure of a complete ortholattice is represented by a global logic, which is called a **predicate orthologic**.

### 5.6.1 Formal system of predicate orthologic OL

The formal system of predicate orthologic, denoted by OL, is obtained from the lattice-valued logic in Titani [13], by adding a primitive logical symbol  $^\perp$  together with rules for ortho-complementation. That is, primitive logical symbols of OL are

$$\wedge, \vee, ^\perp, \rightarrow, \forall, \exists.$$

Defined formulas are:

$$\begin{aligned}
 \top &\stackrel{\text{def}}{\iff} (\varphi \vee \varphi^\perp) \\
 \perp &\stackrel{\text{def}}{\iff} (\top)^\perp \\
 \neg\varphi &\stackrel{\text{def}}{\iff} \varphi \rightarrow \perp \\
 \varphi \leftrightarrow \psi &\stackrel{\text{def}}{\iff} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \\
 \Box\varphi &\stackrel{\text{def}}{\iff} (\varphi \rightarrow \varphi) \rightarrow \varphi \\
 \Diamond\varphi &\stackrel{\text{def}}{\iff} (\Box\varphi^\perp)^\perp
 \end{aligned}$$

The inference rules of predicate orthologic OL are those of lattice valued logic LL plus the following inference rules (C1), (C2), (C3):

$$\begin{aligned}
 (C1) : \quad & \frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi^{\perp\perp}, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi^{\perp\perp}} \\
 (C2) : \quad & \frac{\Gamma \Rightarrow \overline{\Delta}, \varphi}{\varphi^\perp, \Gamma \Rightarrow \overline{\Delta}} \quad \frac{\Gamma \Rightarrow \Delta, \overline{\varphi}}{\overline{\varphi}^\perp, \Gamma \Rightarrow \Delta} \quad \frac{\varphi, \overline{\Gamma} \Rightarrow \Delta}{\overline{\Gamma} \Rightarrow \Delta, \varphi^\perp} \quad \frac{\overline{\varphi}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \overline{\varphi}^\perp} \\
 (C3) : \quad & \frac{\varphi, \overline{\Gamma} \Rightarrow \overline{\Delta}, \psi}{\psi^\perp, \overline{\Gamma} \Rightarrow \overline{\Delta}, \varphi^\perp}
 \end{aligned}$$

If  $\Gamma \Rightarrow \Delta$  is provable in OL, then we write “ $\text{OL} \vdash \Gamma \Rightarrow \Delta$ ”. “ $\text{OL} \vdash \varphi \Rightarrow \psi$ ” and “ $\text{OL} \vdash \psi \Rightarrow \varphi$ ” can be shortened as “ $\text{OL} \vdash \varphi \Leftrightarrow \psi$ ” and also “ $\text{OL} \vdash \Rightarrow \varphi$ ” as “ $\text{OL} \vdash \varphi$ ”.

A formula of the form  $\Box\varphi$  is called a  $\Box$ -closed formula.  $\overline{\varphi}$  expresses that  $\varphi$  is  $\Box$ -closed.

**THEOREM 5.6.1.** *For any formula  $\varphi$  of OL:*

- (1)  $\text{OL} \vdash \Box\varphi \Rightarrow \varphi$  ,  $\text{OL} \vdash \overline{\varphi} \Leftrightarrow \Box\overline{\varphi}$ ;
- (2)  $\text{OL} \vdash \top \Leftrightarrow \Box\top$  ,  $\text{OL} \vdash \perp \Leftrightarrow \Box\perp$ .

*Proof.* (1)  $(\varphi \rightarrow \varphi)$  is  $\Box$ -closed, and  $\text{OL} \vdash (\varphi \rightarrow \varphi)$ . Hence, by  $(\rightarrow)$ :

$$\frac{\Rightarrow (\varphi \rightarrow \varphi) \quad \varphi \Rightarrow \varphi}{((\varphi \rightarrow \varphi) \rightarrow \varphi) \Rightarrow \varphi} \quad \frac{(\overline{\varphi} \rightarrow \overline{\varphi}), \overline{\varphi} \Rightarrow \overline{\varphi}}{\overline{\varphi} \Rightarrow ((\overline{\varphi} \rightarrow \overline{\varphi}) \rightarrow \overline{\varphi})}.$$

(2) By Rule (C2),  $\text{OL} \vdash \varphi \vee \varphi^\perp$  for any formula  $\varphi$ . Hence,

$$\text{OL} \vdash (\overline{\psi} \vee \overline{\psi}^\perp) \Leftrightarrow (\varphi \vee \varphi^\perp) \Leftrightarrow \top \quad \text{and} \quad \text{OL} \vdash (\overline{\psi} \wedge \overline{\psi}^\perp) \Leftrightarrow (\varphi \wedge \varphi^\perp) \Leftrightarrow \perp.$$

□

**THEOREM 5.6.2.** *For formulas  $\varphi, \psi, \xi$  of OL, we have*

- (1)  $\text{OL} \vdash \varphi \wedge \neg\varphi \Rightarrow ; \quad \text{OL} \vdash \Rightarrow \overline{\varphi} \vee \neg\overline{\varphi};$
- (2)  $\text{OL} \vdash \overline{\varphi}^\perp \Leftrightarrow \neg\overline{\varphi};$
- (3)  $\text{OL} \vdash \varphi \Rightarrow \psi$  if and only if  $\text{OL} \vdash (\varphi \rightarrow \psi);$
- (4) If  $\varphi(\psi)$  is a formula with sub-formula  $\psi$  and  $\text{OL} \vdash \psi \Leftrightarrow \xi$ , then

$$\text{OL} \vdash \varphi(\psi) \Leftrightarrow \varphi(\xi).$$

*Proof.* (1)  $\text{OL} \vdash \Rightarrow \top$  and  $\text{OL} \vdash \perp \Rightarrow$  by Rule (C2). Hence, by Rule ( $\supset$ ):

$$\frac{\varphi \Rightarrow \varphi \quad \perp \Rightarrow}{\varphi, (\varphi \rightarrow \perp) \Rightarrow} \quad \frac{\overline{\varphi}, \overline{\varphi}^\perp \Rightarrow \perp}{\overline{\varphi}^\perp \Rightarrow (\overline{\varphi} \rightarrow \perp)}$$

where  $\neg\varphi$  is an abbreviation of  $(\varphi \rightarrow \perp)$ .

- (2) By (1) and (C2).
- (3) By Rule ( $\rightarrow$ ).
- (4) If  $\text{OL} \vdash \varphi \Leftrightarrow \psi$ , then  $\text{OL} \vdash \varphi^\perp \Leftrightarrow \psi^\perp$  by (C3). Hence, (4) holds by 5.4.2.

□

Immediately from (C1), (C2), (C3), we have:

**THEOREM 5.6.3.** *For formulas  $\varphi, \psi$  of OL,*

- (1)  $\text{OL} \vdash \varphi \Leftrightarrow \varphi^{\perp\perp};$
- (2)  $\text{OL} \vdash \Rightarrow \varphi \vee \varphi^\perp, \quad \text{OL} \vdash \varphi \wedge \varphi^\perp \Rightarrow ;$
- (3)  $\text{OL} \vdash (\varphi \rightarrow \psi) \Leftrightarrow (\psi^\perp \rightarrow \varphi^\perp).$

**THEOREM 5.6.4.** *A theorem of lattice-valued logic LL is a theorem of OL. That is,*

$$\text{LL} \vdash \varphi \quad \text{implies} \quad \text{OL} \vdash \varphi.$$

*Proof.* The inference rules of LL are  $(\wedge)$ ,  $(\vee)$ ,  $(\rightarrow)$ ,  $(\forall)$ , and  $(\exists)$  of OL together with the following four rules of  $(\neg)$ . Hence, it suffices to show that if the upper sequent of each rule of  $(\neg)$  is provable in OL, then so is the lower sequent.

$$\neg : \quad \frac{\Gamma \Rightarrow \bar{\Delta}, \varphi}{\neg \varphi, \Gamma \Rightarrow \bar{\Delta}} \quad \frac{\Gamma \Rightarrow \Delta, \bar{\varphi}}{\neg \bar{\varphi}, \Gamma \Rightarrow \Delta} \quad \frac{\varphi, \bar{\Gamma} \Rightarrow \bar{\Delta}}{\bar{\Gamma} \Rightarrow \bar{\Delta}, \neg \varphi} \quad \frac{\bar{\varphi}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \bar{\varphi}},$$

where  $\neg \varphi$  is an abbreviation of  $(\varphi \supset \perp)$ . □

**THEOREM 5.6.5.**

- (1)  $\text{OL} \vdash (\varphi \vee \psi)^\perp \Leftrightarrow (\varphi^\perp \wedge \psi^\perp)$ ,  $\text{OL} \vdash (\varphi^\perp \vee \psi^\perp) \Leftrightarrow (\varphi \wedge \psi)^\perp$ ;
- (2)  $\text{OL} \vdash (\forall x \varphi(x))^\perp \Leftrightarrow \exists x \varphi^\perp(x)$ ,  $\text{OL} \vdash (\exists x \varphi(x))^\perp \Leftrightarrow \forall x \varphi^\perp(x)$ .

*Proof.* Straightforward. □

## 5.6.2 Interpretation of OL

Let  $\mathcal{L}$  be a complete ortholattice and  $M = \langle \mathcal{L}, D, I \rangle$ , where  $D$  is a domain of variables, and  $I$  is an interpretation of predicate symbols

$$I(p^n) : D^n \rightarrow \mathcal{L}.$$

$M = \langle \mathcal{L}, D, I \rangle$  is called a **ortholattice valued model** of OL, where  $\rightarrow$ ,  $\neg$  and  $^\perp$  are interpreted as

$$\begin{aligned} \llbracket \Box \varphi \rrbracket &= \Box \llbracket \varphi \rrbracket = \begin{cases} 1, & \text{if } \llbracket \varphi \rrbracket = 1 \\ 0, & \text{otherwise} \end{cases} \\ \llbracket \neg \varphi \rrbracket &= \neg \llbracket \varphi \rrbracket = \begin{cases} 1, & \text{if } \llbracket \varphi \rrbracket = 0 \\ 0, & \text{otherwise.} \end{cases} \\ \llbracket \varphi^\perp \rrbracket &= (\llbracket \varphi \rrbracket)^\perp. \end{aligned}$$

An ortholattice valued model  $\mathbf{M}$  of OL, shortly OL-valued model, is a lattice valued model. So obviously we have

**THEOREM 5.6.6** (Soundness). *If a sequent  $\Phi \Rightarrow \Psi$  is provable in OL, then it is valid.*

### 5.6.3 Strong completeness of OL

Only difference between LL and OL is that OL has the logical operator  $^\perp$ .

**THEOREM 5.6.7** (Strong completeness). *If a generalized sequent  $\Phi \Rightarrow \Psi$  of OL is valid, where  $\Phi$  and  $\Psi$  are not necessarily finite, then it is provable in OL :*

$$\models \Phi \Rightarrow \Psi \quad \Rightarrow \quad \text{OL} \vdash \Phi \Rightarrow \Psi.$$

*Proof.* The proof of strong completeness of OL is in Titani-Kodera-Aoyama[14], which is similar to the proof for lattice valued logic LL by Takano [10].

Supposing that  $\text{OL} \not\vdash \Phi \Rightarrow \Psi$ , we will construct  $\Phi_\infty$  in the same way as the case of LL, and define a binary relation  $\equiv$  between formulas by

$$A \equiv B \stackrel{\text{def}}{\iff} ((A \rightarrow B) \in \Phi_\infty \quad \text{and} \quad (B \rightarrow A) \in \Phi_\infty)$$

$\equiv$  is an equivalence relation on the set of formulas. Let  $|A|$  denote the equivalence class to which  $A$  belongs:  $|A| \stackrel{\text{def}}{=} \{B \mid A \equiv B\}$ , and let

$$\begin{aligned} P &\stackrel{\text{def}}{=} \{|A| \mid A \text{ is a formula}\} \\ |A| \leq |B| &\stackrel{\text{def}}{\iff} (A \rightarrow B) \in \Phi_\infty \\ 1 &\stackrel{\text{def}}{=} |\top| \\ 0 &\stackrel{\text{def}}{=} |\perp| \end{aligned}$$

$A \equiv B$  if and only if  $A^\perp \equiv B^\perp$ , hence  $|A|^\perp$  is defined by

$$|A|^\perp \stackrel{\text{def}}{=} |A^\perp|.$$

$P$  is a lattice, where

- (1)  $|A \wedge B| = |A| \wedge |B|$ ;
- (2)  $|A \vee B| = |A| \vee |B|$ ;

- (3)  $|A \rightarrow B| = \begin{cases} 1 & \text{if } |A| \leq |B|, \\ 0 & \text{otherwise;} \end{cases}$
- (4)  $|A^\perp| = |A|^\perp$
- (5)  $|\forall x A(x)| = \bigwedge \{|A(a)| \mid a \in FV\};$
- (6)  $|\exists x A(x)| = \bigvee \{|A(a)| \mid a \in FV\}.$
- (7)  $\bigwedge$  and  $\bigvee$  above denote the infimum and the supremum, respectively, in  $P$ .
- (8) For formulas  $A, B$  of OL :
  - (a)  $|A|^{\perp\perp} = |A|;$
  - (b)  $|A| \wedge |A|^\perp = 0, \quad |A| \vee |A|^\perp = 1;$
  - (c)  $|A| \leq |B| \iff |B|^\perp \leq |A|^\perp.$

Thus,  $P = \langle P, \leq, \bigwedge, \bigvee, ^\perp \rangle$  is an ortholattice.

By Mcneille's theorem( cf. Lemma 5.4.9), any poset  $P$  can be embedded in a complete lattice  $\mathcal{L}$ , so that order is preserved, together with the suprema and infima existing in  $P$ , where for each  $X \subset P$ ,

$X^*$  denotes the set of upper bounds in  $P$ , i.e.  $X^* = \{x \in P \mid \forall t \in X (t \leq x)\}$ ,  
 $X^\dagger$  denotes the set of lower bounds in  $P$ , i.e.  $X^\dagger = \{x \in P \mid \forall t \in X (x \leq t)\}$   
 and

$\mathcal{L}$  is defined to be the set of all  $X \subset P$  such that  $X = (X^*)^\dagger$ :

$$\mathcal{L} \stackrel{\text{def}}{=} \{X \subset P \mid X = (X^*)^\dagger\}.$$

Then

$$\bigwedge \{X_i\}_i = \bigcap_i X_i \quad \text{and} \quad \bigvee \{X_i\}_i = ((\bigcup_i X_i)^*)^\dagger \quad \text{for } \{X_i\}_i \subset \mathcal{L}.$$

The embedding  $i : P \rightarrow \mathcal{L}$  is defined by  $i(a) = \{t \in P \mid t \leq a\}$  for  $a \in P$ .

We denote the infimum of  $\{X_i\}_i$  in  $\mathcal{L}$  by  $\bigwedge_i X_i$  and the supremum of  $\{X_i\}_i$  in  $\mathcal{L}$  by  $\bigvee_i X_i$ , i.e.,

$$i(\bigwedge_n t_n) = \bigwedge_n i(t_n) \quad \text{and} \quad i(\bigvee_n t_n) = ((\bigcup_n i(t_n))^*)^\dagger = \bigvee_n i(t_n).$$

**DEFINITION 5.6.1.** For  $X \subset P$ , let

$$X^\perp \stackrel{\text{def}}{=} \{a \in P \mid x \leq a^\perp \text{ for all } x \in X\}.$$

We have

- (1) If  $X \subset P$ , then  $0 \in X^\perp$ ;
- (2) If  $X, Y \subset P$  and  $X \subset Y$ , then  $Y^\perp \subset X^\perp$ ;
- (3) If  $X \subset P$ , then  $X \subset X^{\perp\perp}$  and  $X^\perp = X^{\perp\perp\perp}$ ;
- (4)  $X \cap X^\perp = \{0\}$ ;
- (5)  $(X \cup X^\perp)^{\perp\perp} = P$ .

*Proof.* (1)  $\forall x \in X (0 \leq x^\perp)$ , hence  $0 \in X^\perp$ .

(2) Obvious.

(3) If  $x \in X$  then  $\forall t \in X^\perp (x \leq t^\perp)$ . Therefore,  $X \subset X^{\perp\perp}$ .  
By (2),  $X^\perp \subset X^{\perp\perp\perp}$  and  $X^{\perp\perp\perp} \subset X^\perp$ .

(4) If  $x \in X \cap X^\perp$ , then  $x \leq x^\perp$ , hence  $x = 0$ .

(5)  $(X \cup X^\perp)^\perp \subset X^\perp \cap X^{\perp\perp} = \{0\}$ . Therefore,  $(X \cup X^\perp)^{\perp\perp} = P$ .

□

If  $X \subset P$ , then  $(X^*)^\dagger = X^{\perp\perp}$ .

*Proof.*

$$\begin{aligned} z \in (X^*)^\dagger &\iff \forall y \left( \forall x \in X (x \leq y) \Rightarrow z \leq y \right) \\ &\iff \forall y \left( y^\perp \in X^\perp \Rightarrow y^\perp \leq z^\perp \right) \\ &\iff \forall y \left( y^\perp \in X^\perp \Rightarrow z \leq (y^\perp)^\perp \right) \iff z \in X^{\perp\perp} \end{aligned}$$

□

Thus,  $P$  is embedded in the complete ortholattice

$$\mathcal{L} = \{X \subset P \mid X = X^{\perp\perp}\} \quad \text{by} \quad i : a \mapsto i(a) = \{t \in P \mid t \leq a\} \quad \text{for } a \in P,$$

where

$$\begin{aligned} i(\bigwedge_n t_n) &= \bigwedge_n i(t_n) = \bigcap_n i(t_n), & i(\bigvee_n t_n) &= \bigvee_n i(t_n) = (\bigcup_n i(t_n))^{\perp\perp}, \\ i(a^\perp) &= i(a)^\perp. \end{aligned}$$

Let  $D = FV$  and  $I(p) : D^n \rightarrow \mathcal{L}$  for  $n$ -ary predicate constant  $p$  be defined by

$$I(p)(a_1, \dots, a_n) = i(|p(a_1, \dots, a_n)|) \quad \text{for every } a_1, \dots, a_n \in D.$$

Then  $\mathbf{M} = \langle \mathcal{L}, D, I \rangle$  is an OL-valued model and For every formula  $A$ ,  $A[\mathbf{M}, v] = i(|A^v|)$ , where  $A^v$  is the formula obtained from  $A$  by replacing every occurrence of each  $a \in FV$  by the free variable  $v(a)$ .

Let  $u$  be the  $D$ -assignment which is the identity map on  $D$ .

$$A[\mathbf{M}, u] = i(|A|) \quad \text{for every } A.$$

The proof of completeness of OL is completed with the following proposition.

$$\bigwedge \{A[\mathbf{M}, u] \mid A \in \Phi\} \not\leq \bigvee \{B[\mathbf{M}, u] \mid B \in \Psi\}. \quad (5.6.1)$$

*Proof.* It suffices to show that

$$|p| \in \bigcap \{i(|A|) \mid A \in \Phi\}; \quad (5.6.2)$$

$$|p| \notin (\bigcup \{i(|B|) \mid B \in \Psi\})^{\perp\perp}. \quad (5.6.3)$$

Since  $|p| \leq |A|$  for every  $A \in \Phi$ , we have (5.6.2).

Now we assume  $|p| \in (\bigcup \{i(|B|) \mid B \in \Psi\})^{\perp\perp}$ . Then

$$|p| \leq |X|^\perp \quad \text{for all } |X| \in (\bigcup_{B \in \Psi} (i(|B|))^\perp = \bigcap_{B \in \Psi} (i(|B|)))^\perp.$$

Since  $|B| \leq |q|$  for all  $B \in \Psi$ , we have

$$|X| \leq |q|, \quad \text{whenever } |X| \leq |B|.$$

$$\begin{aligned} \therefore |q|^\perp &\in \bigcap_{B \in \Psi} \{|X| \mid |X| \leq |B|\}^\perp = \bigcap_{B \in \Psi} (i(|B|))^\perp. \\ \therefore |p| &\leq |q|^{\perp\perp} = |q|. \quad \therefore (p \rightarrow q) \in \Phi_\infty, \end{aligned}$$

which is a contradiction. Hence we have equation (5.6.1). □

□

## 5.7 Quantum Logic

The mathematical language of lattice was introduced into quantum physics by Birkhoff and von Neumann [2].

The truth value set of the standard quantum theory is expressed by the lattice  $Q(\mathcal{H})$  with automorphisms, where  $\mathcal{H}$  is a Hilbert space  $\mathcal{H}$ , and  $Q(\mathcal{H})$  is a complete orthomodular lattice 3.6.2 consisting of all projections (or equivalently all closed subspaces) of  $\mathcal{H}$ . Thus, the logic of standard quantum theory is represented by the complete orthomodular lattice  $Q(\mathcal{H})$ .

**Orthomodular lattice** 3.6.2 is an ortholattice 3.5.1 satisfying orthomodularity (P), i.e. a lattice provided with an operator  $^\perp$  satisfying

$$(C1) \quad a^{\perp\perp} = a;$$

$$(C2) \quad a \vee a^\perp = 1, \quad a \wedge a^\perp = 0;$$

$$(C3) \quad a \leq b \implies b^\perp \leq a^\perp;$$

$$(P) \quad a \leq b \implies b = a \vee (b \wedge a^\perp) \quad (\text{orthomodularity}).$$

The complete orthomodular lattice is represented by a sheaf of complete Boolean algebra. That is, each formula of quantum logic is interpreted as a cross-section of a sheaf whose stalks are complete Boolean algebras.

The **quantum logic** represents the structure of the complete orthomodular lattices.

G.Takeuti adopted an implication defined in terms of  $\wedge$ ,  $\vee$ , and  $^\perp$ , on the complete orthomodular lattice  $Q(\mathcal{H})$ , in order to develop a quantum set theory. We denote his implication by  $\rightarrow_{\mathbf{T}}$ :

$$(a \rightarrow_{\mathbf{T}} b) \stackrel{\text{def}}{=} a^\perp \vee (a \wedge b).$$

$\rightarrow_{\mathbf{T}}$  is considered as a **local implication**, in the sense that

$$c \leq (a \rightarrow_{\mathbf{T}} b) \quad \text{if and only if} \quad (c \wedge a) \leq (c \wedge b).$$

However, this implication is not enough to develop a global set theory, because it is not transitive. That is,

$$(a \rightarrow_{\mathbf{T}} b) \wedge (b \rightarrow_{\mathbf{T}} c) \not\leq (a \rightarrow_{\mathbf{T}} c).$$

The transitivity of the corresponding logical implication is indispensable for the development of set theory, since equality axioms of set theory which depend on the transitivity of implication are fundamental.

Quantum logic QL is obtained from OL by adding inference rule

$$(P) : \quad \frac{\varphi, \bar{\Gamma} \Rightarrow \bar{\Delta}, \psi}{\psi, \bar{\Gamma} \Rightarrow \bar{\Delta}, \varphi \vee (\psi \wedge \varphi^{\perp})} \quad [\text{Orthomodularity}]$$

i.e.

$$\text{QL} = \text{OL} + (P).$$

QL is a global logic as well as lattice-valued logic LL (Titani [13]) and ortho-lattice OL (Titani [15]) introducing the basic implication  $\rightarrow$ .

Since orthomodular lattice has Takeuti's implication defined by

$$(a \rightarrow_{\mathbf{T}} b) \stackrel{\text{def}}{\iff} a^{\perp} \vee (a \wedge b),$$

the basic implication  $\rightarrow$  is express in terms of the implication  $\rightarrow_{\mathbf{T}}$  and the modal operator  $\Box$  :

$$(a \rightarrow b) = \Box(a \rightarrow_{\mathbf{T}} b).$$

The logical system provided with the globalization  $\Box$  instead of the basic implication is denoted by  $\text{QL}_{\Box}$ .  $\text{QL}_{\Box}$ , which is equivalent to QL, had been introduced in Titani-Kodera-Aoyama [14].

**Primitive logical symbols** of  $\text{QL}_{\Box}$  are:

$$\wedge, \vee, ^{\perp}, \exists, \forall, \Box.$$

$\Box$  is interpreted on a orthomodular lattice as

$$\Box a = \begin{cases} 1 & a = 1 \\ 0 & \text{otherwise} \end{cases}$$

Finite sequences of formulas are denoted by  $\Gamma, \Delta, \dots$ . If  $\Gamma$  is a sequence “ $\varphi_1, \dots, \varphi_n$ ” of formulas, then the sequence “ $\Box\varphi_1, \dots, \Box\varphi_n$ ” is denoted by  $\Box\Gamma$ .

**Logical axioms of  $QL_{\Box}$**  are sequents of the form  $\varphi \Rightarrow \varphi$ .

**Structural rules :**

|               |                                                                                                                                     |                                                                                                                                     |
|---------------|-------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------|
| Thinning :    | $\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$                                                              | $\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$                                                              |
| Contraction : | $\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$                                            | $\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$                                            |
| Interchange : | $\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta}$                               | $\frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda}$                       |
| Cut :         | $\frac{\Gamma \Rightarrow \Box\Delta, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Box\Delta, \Lambda}$ | $\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Box\Pi \Rightarrow \Lambda}{\Gamma, \Box\Pi \Rightarrow \Delta, \Lambda}$ |
|               | $\frac{\Gamma \Rightarrow \Delta, \Box\varphi \quad \Box\varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$ |                                                                                                                                     |

**Logical rules:**

|            |                                                                                             |                                                                                                                                                   |
|------------|---------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------|
| $\wedge$ : | $\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$ | $\frac{\Gamma \Rightarrow \Box\Delta, \varphi \quad \Gamma \Rightarrow \Box\Delta, \psi}{\Gamma \Rightarrow \Box\Delta, \varphi \wedge \psi}$     |
|            | $\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$    | $\frac{\Gamma \Rightarrow \Delta, \Box\varphi \quad \Gamma \Rightarrow \Delta, \Box\psi}{\Gamma \Rightarrow \Delta, \Box\varphi \wedge \Box\psi}$ |

$$\begin{array}{l}
 \vee : \quad \frac{\varphi, \Box \Gamma \Rightarrow \Delta \quad \psi, \Box \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Box \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \\
 \qquad \frac{\Box \varphi, \Gamma \Rightarrow \Delta \quad \Box \psi, \Gamma \Rightarrow \Delta}{\Box \varphi \vee \Box \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \\
 \forall : \quad \frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Box \Delta, \varphi(a)}{\Gamma \Rightarrow \Box \Delta, \forall x \varphi(x)} \qquad \frac{\Gamma \Rightarrow \Delta, \Box \varphi(a)}{\Gamma \Rightarrow \Delta, \Box \forall x \Box \varphi(x)} \\
 \text{where } t \text{ is any term} \qquad \text{where } a \text{ is a free variable which does} \\
 \qquad \qquad \qquad \text{not occur in the lower sequent.} \\
 \exists : \quad \frac{\varphi(a), \Box \Gamma \Rightarrow \Delta}{\exists x \varphi(x), \Box \Gamma \Rightarrow \Delta} \quad \frac{\Box \varphi(a), \Gamma \Rightarrow \Delta}{\Diamond \exists x \Box \varphi(x), \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)} \\
 \text{where } a \text{ is a free variable which does} \qquad \text{where } t \text{ is any term} \\
 \qquad \qquad \qquad \text{not occur in the lower sequent.} \\
 (C1) : \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi^{\perp\perp}, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi^{\perp\perp}} \\
 (C2) : \quad \frac{\Gamma \Rightarrow \Box \Delta, \varphi}{\varphi^{\perp}, \Gamma \Rightarrow \Box \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \Box \varphi}{(\Box \varphi)^{\perp}, \Gamma \Rightarrow \Delta} \quad \frac{\varphi, \Box \Gamma \Rightarrow \Delta}{\Box \Gamma \Rightarrow \Delta, \varphi^{\perp}} \quad \frac{\Box \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, (\Box \varphi)^{\perp}} \\
 (C3) : \quad \frac{\varphi, \Box \Gamma \Rightarrow \Box \Delta, \psi}{\psi^{\perp}, \Box \Gamma \Rightarrow \Box \Delta, \varphi^{\perp}} \\
 (P) : \quad \frac{\varphi, \Box \Gamma \Rightarrow \Box \Delta, \psi}{\psi, \Box \Gamma \Rightarrow \Box \Delta, \varphi \vee (\psi \wedge \varphi^{\perp})} \quad [\text{Orthomodularity}] \\
 (\Box) : \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box \varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Box \Gamma \Rightarrow \Box \Delta, \varphi}{\Box \Gamma \Rightarrow \Box \Delta, \Box \varphi} \quad [\text{Globalization}].
 \end{array}$$

### Remark

Our formal systems LL, OL, GLK, GLJ, QL, QL<sub>□</sub> are all Gentzen-type sequent calculi, provided with the basic implication  $\rightarrow$  or the modal operator  $\Box$ , where cut rules are not eliminable. To have the basic implication  $\rightarrow$  or the modal operator  $\Box$  means that the logical system has a dual structure, consisting of a logic and its meta-logic. The meta-logic is classical.

**Defined formulas** are :

$$\begin{aligned}
 \top &\stackrel{\text{def}}{\iff} (\varphi \vee \varphi^\perp) \\
 \perp &\stackrel{\text{def}}{\iff} (\top)^\perp \\
 \varphi \rightarrow_{\mathbf{T}} \psi &\stackrel{\text{def}}{\iff} \varphi^\perp \vee (\varphi \wedge \psi) \quad (\text{cf. Takeuti}[11],[12]) \\
 \varphi \rightarrow \psi &\stackrel{\text{def}}{\iff} \Box(\varphi \rightarrow_{\mathbf{T}} \psi) \\
 \neg \varphi &\stackrel{\text{def}}{\iff} \varphi \rightarrow \perp \\
 \varphi \leftrightarrow \psi &\stackrel{\text{def}}{\iff} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \\
 \Diamond \varphi &\stackrel{\text{def}}{\iff} (\Box \varphi^\perp)^\perp \\
 \varphi \circ \psi &\stackrel{\text{def}}{\iff} \Box((\varphi \wedge \psi) \vee (\varphi \wedge \psi^\perp) \vee (\varphi^\perp \wedge \psi) \vee (\varphi^\perp \wedge \psi^\perp))
 \end{aligned}$$

**THEOREM 5.7.1.** *For formulas  $\varphi, \psi, \xi$  of  $\text{QL}_\Box$ , we have*

- (1)  $\text{QL}_\Box \vdash \Box \varphi \Rightarrow \varphi, \text{QL}_\Box \vdash \varphi \Rightarrow \Diamond \varphi$ ;
- (2)  $\text{QL}_\Box \vdash \Box \Gamma \Rightarrow \Box \Delta, \varphi$  if and only if  $\text{QL}_\Box \vdash \Box \Gamma \Rightarrow \Box \Delta, \Box \varphi$ ;
- (3)  $\text{QL}_\Box \vdash \varphi, \Box \Gamma \Rightarrow \Box \Delta$  if and only if  $\text{QL}_\Box \vdash \Diamond \varphi, \Box \Gamma \Rightarrow \Box \Delta$ ;
- (4)  $\text{QL}_\Box \vdash \Box \varphi \Leftrightarrow \Box \Box \varphi, \text{QL}_\Box \vdash \Diamond \varphi \Leftrightarrow \Box \Diamond \varphi$ ;
- (5)  $\text{QL}_\Box \vdash (\Box \varphi \wedge \Box \psi) \Leftrightarrow \Box(\Box \varphi \wedge \Box \psi)$ ;
- (6)  $\text{QL}_\Box \vdash (\Box \varphi \vee \Box \psi) \Leftrightarrow \Box(\Box \varphi \vee \Box \psi)$ ;
- (7)  $\text{QL}_\Box \vdash (\Box \varphi)^\perp \Leftrightarrow \Box((\Box \varphi)^\perp)$ ;
- (8)  $\text{QL}_\Box \vdash \forall x \Box \varphi(x) \Leftrightarrow \Box \forall x \Box \varphi(x) \Leftrightarrow \Box \forall x \varphi(x)$ ;
- (9)  $\text{QL}_\Box \vdash \exists x \Box \varphi(x) \Leftrightarrow \Box \exists x \Box \varphi(x)$ ;
- (10)  $\text{QL}_\Box \vdash \exists x \Diamond \varphi(x) \Leftrightarrow \Box \exists x \Diamond \varphi(x) \Leftrightarrow \Diamond \exists x \varphi(x)$ .

*Proof.* Straightforward. □

**COROLLARY 5.7.2.** *If  $\varphi$  is a  $\Box$ -closed formula of  $\text{QL}$ , then  $\text{QL}_\Box \vdash \varphi \Leftrightarrow \Box \varphi$ .*

*Proof.* For a formula of the form  $(\varphi \rightarrow \psi)$ ,  $\text{QL}_\Box \vdash (\varphi \rightarrow \psi) \Leftrightarrow \Box(\varphi \rightarrow \psi)$ , since  $\varphi \rightarrow \psi$  is the abbreviation of  $\Box(\varphi^\perp \vee (\varphi \wedge \psi))$ .  $\Box$ -closed formulas of  $\text{QL}$  are constructed from formulas of the form  $\varphi \rightarrow \psi$  by  $\wedge, \vee, \perp, \forall$ , and  $\exists$ . Hence, by Theorem 4.1, we have  $\text{QL}_\Box \vdash \varphi \Leftrightarrow \Box \varphi$ . □

**THEOREM 5.7.3.**

- (1)  $QL_{\Box} \vdash \varphi \circ \psi$ ,  $QL_{\Box} \vdash \psi \circ \varphi$  and  $QL_{\Box} \vdash \varphi \circ \psi^{\perp}$  are all equivalent ;
- (2) If  $QL_{\Box} \vdash \varphi \circ \psi$ , then  $QL_{\Box} \vdash \varphi \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp})$ ;
- (3) If  $QL_{\Box} \vdash \varphi \Rightarrow \psi$ , then  $QL_{\Box} \vdash \varphi \Rightarrow \psi \wedge (\varphi \vee \psi^{\perp})$ ;
- (4) If  $QL_{\Box} \vdash \varphi \circ \psi$ , then  $QL_{\Box} \vdash \varphi \vee \psi^{\perp} \Leftrightarrow (\varphi \wedge \psi) \vee \psi^{\perp}$ .

*Proof.*

- (1) Immediate from the definition of  $\circ$ .

- (2) We have

$$QL_{\Box} \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp}) \Rightarrow \varphi \text{ and } QL_{\Box} \vdash \varphi \Rightarrow (\varphi \vee \psi) \wedge (\varphi \vee \psi^{\perp}). \quad (5.7.1)$$

$$\therefore QL_{\Box} \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp}) \Rightarrow (\varphi \vee \psi) \wedge (\varphi \vee \psi^{\perp}). \quad (5.7.2)$$

By Rule (P) of orthomodularity,

$$QL_{\Box} \vdash (\varphi \vee \psi) \wedge (\varphi \vee \psi^{\perp}) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp}) \vee \left( (\varphi \vee \psi) \wedge (\varphi \vee \psi^{\perp}) \wedge (\varphi^{\perp} \vee \psi^{\perp}) \wedge (\varphi^{\perp} \vee \psi) \right).$$

If  $QL_{\Box} \vdash \varphi \circ \psi$ , then

$$QL_{\Box} \vdash (\varphi \vee \psi) \wedge (\varphi \vee \psi^{\perp}) \wedge (\varphi^{\perp} \vee \psi^{\perp}) \wedge (\varphi^{\perp} \vee \psi) \Rightarrow \perp.$$

$$\therefore QL_{\Box} \vdash (\varphi \vee \psi) \wedge (\varphi \vee \psi^{\perp}) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp}).$$

$$\therefore QL_{\Box} \vdash \varphi \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp}) \quad \text{by (5.7.1) and (5.7.2).}$$

- (3) If  $QL_{\Box} \vdash \varphi \Rightarrow \psi$ , then  $QL_{\Box} \vdash \psi^{\perp} \Rightarrow \varphi^{\perp}$ . By Rule (P),  $QL_{\Box} \vdash \varphi^{\perp} \Leftrightarrow \psi^{\perp} \vee (\varphi^{\perp} \wedge \psi)$ . Hence,  $QL_{\Box} \vdash \varphi \Leftrightarrow \psi \wedge (\varphi \vee \psi^{\perp})$ .

- (4) We have  $QL_{\Box} \vdash (\varphi \wedge \psi) \vee \psi^{\perp} \Rightarrow \varphi \vee \psi^{\perp}$ . Hence, by (3),

$$QL_{\Box} \vdash (\varphi \wedge \psi) \vee \psi^{\perp} \Rightarrow (\varphi \vee \psi^{\perp}) \wedge \left( (\varphi \wedge \psi) \vee \psi^{\perp} \right) \vee (\varphi \vee \psi^{\perp})^{\perp}.$$

By  $QL_{\Box} \vdash \varphi \circ \psi$  and (2),

$$QL_{\Box} \vdash (\varphi \wedge \psi) \vee \psi^{\perp} \vee (\varphi \vee \psi^{\perp})^{\perp} \Leftrightarrow \psi \vee \psi^{\perp} \Leftrightarrow \top.$$

$$\therefore QL_{\Box} \vdash (\varphi \wedge \psi) \vee \psi^{\perp} \Leftrightarrow \varphi \vee \psi^{\perp}.$$

□

**THEOREM 5.7.4.**

- (1)  $QL_{\Box} \vdash \varphi \circ \psi$  if and only if  $QL_{\Box} \vdash \varphi \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp})$ ;
- (2)  $QL_{\Box} \vdash \varphi \circ \psi$  if and only if  $QL_{\Box} \vdash (\varphi \vee \psi^{\perp}) \wedge \psi \Leftrightarrow \varphi \wedge \psi$ ;
- (3) If  $QL_{\Box} \vdash \varphi \Rightarrow \psi$ , then  $QL_{\Box} \vdash \varphi \circ \psi$ .

*Proof.* (1) Assume  $QL_{\Box} \vdash \varphi \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp})$ . Then

$$QL_{\Box} \vdash \psi \wedge \varphi^{\perp} \Leftrightarrow \psi \wedge (\varphi^{\perp} \vee \psi^{\perp}) \wedge (\varphi^{\perp} \vee \psi). \\ \therefore QL_{\Box} \vdash \psi \wedge \varphi^{\perp} \Leftrightarrow \psi \wedge (\varphi^{\perp} \vee \psi^{\perp}). \quad (5.7.3)$$

Applying Rule (P) to  $QL_{\Box} \vdash \varphi \wedge \psi \Rightarrow \psi$ ,

$$QL_{\Box} \vdash \psi \Leftrightarrow (\varphi \wedge \psi) \vee (\psi \wedge (\varphi^{\perp} \vee \psi^{\perp})).$$

$$\text{By (5.7.3), } QL_{\Box} \vdash \psi \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi^{\perp} \wedge \psi). \quad (5.7.4)$$

Using the assumption and (5.7.4),

$$QL_{\Box} \vdash \varphi \vee \psi \vee (\varphi \vee \psi)^{\perp} \Leftrightarrow \varphi \vee \psi \vee (\varphi^{\perp} \wedge \psi^{\perp}) \Leftrightarrow \\ (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp}) \vee (\varphi^{\perp} \wedge \psi) \vee (\varphi^{\perp} \wedge \psi^{\perp}). \\ \therefore QL_{\Box} \vdash (\varphi \wedge \psi) \vee (\varphi^{\perp} \wedge \psi) \vee (\varphi \wedge \psi^{\perp}) \vee (\varphi^{\perp} \wedge \psi^{\perp}).$$

It follows that  $QL_{\Box} \vdash \varphi \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp})$  implies  $QL_{\Box} \vdash \varphi \circ \psi$ .

The converse is Theorem 5.7.3(2).

- (2) Assume  $QL_{\Box} \vdash (\varphi \vee \psi^{\perp}) \wedge \psi \Leftrightarrow \varphi \wedge \psi$ . Then

$$QL_{\Box} \vdash (\varphi \wedge \psi) \vee (\varphi^{\perp} \wedge \psi) \Leftrightarrow ((\varphi \vee \psi^{\perp}) \wedge \psi) \vee (\varphi^{\perp} \wedge \psi). \quad (5.7.5)$$

By  $QL_{\Box} \vdash (\varphi^{\perp} \wedge \psi) \Rightarrow \psi$  and Rule (P),

$$QL_{\Box} \vdash (\varphi^{\perp} \wedge \psi) \vee ((\varphi \vee \psi^{\perp}) \wedge \psi) \Leftrightarrow B.$$

$$\text{By (5.7.5) } QL_{\Box} \vdash (\varphi \wedge \psi) \vee (\varphi^{\perp} \wedge \psi) \Leftrightarrow \psi.$$

Therefore,  $QL_{\Box} \vdash \varphi \circ \psi$  by (1).

(3) If  $QL_{\Box} \vdash \varphi \Rightarrow \psi$ , then

$$QL_{\Box} \vdash \psi \Rightarrow \varphi \vee (\psi \wedge \varphi^{\perp}) \quad \text{by (P)} \quad \text{and} \quad QL_{\Box} \vdash \varphi \Leftrightarrow \varphi \wedge \psi.$$

Therefore,  $QL_{\Box} \vdash \psi \Rightarrow (\varphi \wedge \psi) \vee (\psi \wedge \varphi^{\perp})$ .

Hence,  $QL_{\Box} \vdash \varphi \circ \psi$  by (1).

□

**THEOREM 5.7.5.**

(1) If  $QL_{\Box} \vdash (\varphi \circ \psi) \wedge (\varphi \circ \xi)$ , then

$$QL_{\Box} \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \xi) \Leftrightarrow \varphi \wedge (\psi \vee \xi) \quad \text{and}$$

$$QL_{\Box} \vdash (\varphi \vee \psi) \wedge (\varphi \vee \xi) \Leftrightarrow \varphi \vee (\psi \wedge \xi).$$

(2) If  $QL_{\Box} \vdash \varphi \circ \psi(a)$ , then

$$QL_{\Box} \vdash \exists x(\varphi \wedge \psi(x)) \Leftrightarrow \varphi \wedge \exists x\psi(x) \quad \text{and}$$

$$QL_{\Box} \vdash \forall x(\varphi \vee \psi(x)) \Leftrightarrow \varphi \vee \forall x\psi(x).$$

*Proof.* (1) Assume  $QL_{\Box} \vdash (\varphi \circ \psi) \wedge (\varphi \circ \xi)$ . We have

$$QL_{\Box} \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \xi) \Rightarrow \varphi \wedge (\psi \vee \xi).$$

By Rule (P),

$$QL_{\Box} \vdash \varphi \wedge (\psi \vee \xi) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \xi) \vee \\ \left( \varphi \wedge (\psi \vee \xi) \wedge (\varphi^{\perp} \vee \psi^{\perp}) \wedge (\varphi^{\perp} \vee \xi^{\perp}) \right).$$

By Theorem 4.4(2),

$$QL_{\Box} \vdash \varphi \wedge (\varphi^{\perp} \vee \psi^{\perp}) \Leftrightarrow \varphi \wedge \psi^{\perp} \quad \text{and}$$

$$QL_{\Box} \vdash \varphi \wedge (\varphi^{\perp} \vee \xi^{\perp}) \Leftrightarrow \varphi \wedge \xi^{\perp}$$

$$\therefore QL_{\Box} \vdash \varphi \wedge (\psi \vee \xi) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \xi) \vee \left( \varphi \wedge (\psi \vee \xi) \wedge \psi^{\perp} \wedge \xi^{\perp} \right).$$

Since  $QL_{\Box} \vdash \varphi \wedge (\psi \vee \xi) \wedge \psi^{\perp} \wedge \xi^{\perp} \Rightarrow \perp$ ,

$$QL_{\Box} \vdash \varphi \wedge (\psi \vee \xi) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \xi).$$

We have  $QL_{\Box} \vdash \varphi^{\perp} \circ \psi^{\perp}$  and  $QL_{\Box} \vdash \varphi^{\perp} \circ \xi^{\perp}$ . Therefore,

$$QL_{\Box} \vdash (\varphi \vee \psi) \wedge (\varphi \vee \xi) \Leftrightarrow \varphi \vee (\psi \wedge \xi).$$

(2) Similarly, assume  $\text{QL}_\square \vdash \varphi \circ \psi(a)$ . We have

$$\text{QL}_\square \vdash \exists x(\varphi \wedge \psi(x)) \Rightarrow \varphi \wedge \exists x\psi(x).$$

By Rule (P),

$$\text{QL}_\square \vdash \varphi \wedge \exists x\psi(x) \Leftrightarrow \exists x(\varphi \wedge \psi(x)) \vee \left( \varphi \wedge \exists x\psi(x) \wedge \forall x(\varphi^\perp \vee \psi^\perp(x)) \right).$$

Since  $\text{QL}_\square \vdash \varphi \circ \psi(a)$ , we have

$$\begin{aligned} \text{QL}_\square \vdash \varphi \wedge \exists x\psi(x) \wedge \forall x(\varphi^\perp \vee \psi^\perp(x)) \\ \Leftrightarrow \varphi \wedge \exists x\psi(x) \wedge \forall x \left( \varphi \wedge (\varphi^\perp \vee \psi^\perp(x)) \right) \\ \Leftrightarrow \varphi \wedge \exists x\psi(x) \wedge \forall x(\varphi \wedge \psi^\perp(x)), \text{ by Theorem 4.4(2)} \\ \Leftrightarrow \varphi \wedge \exists x\psi(x) \wedge \forall x\psi^\perp(x) \Leftrightarrow \perp \end{aligned}$$

$$\therefore \text{QL}_\square \vdash \varphi \wedge \exists x\psi(x) \Leftrightarrow \exists x(\varphi \wedge \psi(x)).$$

□

**THEOREM 5.7.6.**

- (1)  $\text{QL}_\square \vdash \varphi, \varphi^\perp \vee (\varphi \wedge \psi) \Rightarrow \psi$ . That is,  $\text{QL}_\square \vdash \varphi, \varphi \rightarrow_{\mathbf{T}} \psi \Rightarrow \psi$ . It follows that  $\rightarrow_{\mathbf{T}}$  represents an implication.
- (2)  $\text{QL}_\square \vdash \Box \Gamma, \varphi \Rightarrow \psi$  if and only if  $\text{QL}_\square \vdash \Box \Gamma \Rightarrow \Box(\varphi \rightarrow_{\mathbf{T}} \psi)$ . Therefore, the following definition of the basic implication in  $\text{QL}_\square$  is justified :

$$\varphi \rightarrow \psi \stackrel{\text{def}}{\iff} \Box(\varphi \rightarrow_{\mathbf{T}} \psi).$$

*Proof.* (1) We have  $\text{QL}_\square \vdash \varphi \circ \varphi^\perp$  and  $\text{QL}_\square \vdash \varphi \circ (\varphi \wedge \psi)$ . Hence, by Theorem 5.7.5,

$$\begin{aligned} \text{QL}_\square \vdash \varphi \wedge \left( \varphi^\perp \vee (\varphi \wedge \psi) \right) &\Rightarrow (\varphi \wedge \varphi^\perp) \vee (\varphi \wedge \psi) \\ &\Rightarrow \psi. \end{aligned}$$

(2) If  $QL_{\Box} \vdash \Box\Gamma, \varphi \Rightarrow \psi$ , then  $QL_{\Box} \vdash \Box\Gamma, \varphi \Rightarrow \varphi \wedge \psi$ . Hence

$$QL_{\Box} \vdash \Box\Gamma, \varphi^{\perp} \vee \varphi \Rightarrow \varphi^{\perp} \vee (\varphi \wedge \psi). \quad \therefore QL_{\Box} \vdash \Box\Gamma \Rightarrow \Box(\varphi \rightarrow_{\mathbf{T}} \psi).$$

Conversely, if  $QL_{\Box} \vdash \Box\Gamma \Rightarrow \Box(\varphi \rightarrow_{\mathbf{T}} \psi)$ , then

$$QL_{\Box} \vdash \Box\Gamma, \varphi \Rightarrow \varphi \wedge (\varphi \rightarrow_{\mathbf{T}} \psi).$$

By (1), we have  $QL_{\Box} \vdash \Box\Gamma, \varphi \Rightarrow \psi$ .

□

**THEOREM 5.7.7.**  $QL_{\Box} \vdash \Box\varphi \circ \psi$  for any formulas  $\varphi$  and  $\psi$ .

*Proof.* By using Rule ( $\vee$ )-left,

$$QL_{\Box} \vdash \Box\varphi \Leftrightarrow \Box\varphi \wedge (\psi \vee \psi^{\perp}) \Leftrightarrow (\Box\varphi \wedge \psi) \vee (\Box\varphi \wedge \psi^{\perp}).$$

Therefore, by Theorem 5.7.4,  $QL_{\Box} \vdash \Box\varphi \circ \psi$ .

□

### 5.7.1 Relation between LL, OL, QL and $QL_{\Box}$

By Theorem 3.4,  $LL \vdash \varphi$  implies  $OL \vdash \varphi$ , and QL is  $OL + (P)$ . We show in this section that  $QL_{\Box}$  is equivalent to QL, and also  $GLJ_{\Box}$  is equivalent to GLJ.

$(\varphi \rightarrow \psi)$  in  $QL_{\Box}$  is the abbreviation of  $\Box(\varphi^{\perp} \vee (\varphi \wedge \psi))$ , and  $\Box\psi$  in QL is the abbreviation of  $((\psi \rightarrow \psi) \rightarrow \psi)$ .

**THEOREM 5.7.8.**  $QL \vdash \varphi$  implies  $QL_{\Box} \vdash \varphi$ .

*Proof.* If  $\varphi$  is a  $\Box$ -closed formula of QL, then  $QL_{\Box} \vdash \varphi \Leftrightarrow \Box\varphi$  by Corollary 4.2. Hence, it suffices to show that if the upper sequent(s) of each of the following rules of QL is provable in  $QL_{\Box}$ , then so is the lower sequent:

$$\rightarrow: \frac{\Gamma \Rightarrow \Box\Delta, \varphi \quad \psi, \Box\Pi \Rightarrow \Lambda}{(\varphi \rightarrow \psi), \Gamma, \Box\Pi \Rightarrow \Box\Delta, \Lambda} \quad \frac{\varphi, \Box\Gamma \Rightarrow \Box\Delta, \psi}{\Box\Gamma \Rightarrow \Box\Delta, (\varphi \rightarrow \psi)} \quad \frac{\Box\varphi, \Gamma \Rightarrow \Delta, \Box\psi}{\Gamma \Rightarrow \Delta, (\Box\varphi \rightarrow \Box\psi)}.$$

(1) If  $QL_{\Box} \vdash \Gamma \Rightarrow \Box\Delta, \varphi$  and  $QL_{\Box} \vdash \psi, \Box\Pi \Rightarrow \Lambda$ , then by (Cut) together with  $QL_{\Box} \vdash \varphi, (\varphi \rightarrow \psi) \Rightarrow \psi$ ,

$$QL_{\Box} \vdash (\varphi \rightarrow \psi), \Gamma, \Box\Pi \Rightarrow \Box\Delta, \Lambda.$$

(2)  $QL_{\Box} \vdash \varphi, \Box\Gamma \Rightarrow \Box\Delta, \psi$  implies  $QL_{\Box} \vdash \Box\Gamma \Rightarrow \Box\Delta, (\varphi \rightarrow \psi)$ , by Theorem 5.7.6(2).

(3) If  $QL_{\Box} \vdash \Box\varphi, \Gamma \Rightarrow \Delta, \Box\psi$ , then  $QL_{\Box} \vdash \Box\varphi, \Gamma \Rightarrow \Delta, \Box\varphi \wedge \Box\psi$ . Hence,

$$QL_{\Box} \vdash (\Box\varphi)^{\perp} \vee \Box\varphi, \Gamma \Rightarrow \Delta, (\Box\varphi)^{\perp} \vee (\Box\varphi \wedge \Box\psi).$$

Therefore,  $QL_{\Box} \vdash \Box\varphi, \Gamma \Rightarrow \Delta, \Box\psi$  implies  $QL_{\Box} \vdash \Gamma \Rightarrow \Delta, (\Box\varphi \rightarrow \Box\psi)$ .

□

**THEOREM 5.7.9.**  $QL_{\Box} \vdash \varphi$  implies  $QL \vdash \varphi$ .

*Proof.*  $\Box\varphi$  is an abbreviation of  $((\varphi \rightarrow \varphi) \rightarrow \varphi)$ , hence  $\Box\varphi$  is  $\Box$ -closed.  $\Box\Gamma, \Box\Delta, \dots$  are sequences of  $\Box$ -closed formulas. Hence, it suffices to show that if the upper sequent of each of the following rules of  $QL_{\Box}$  is provable in  $QL$ , then so is the lower sequent:

$$(\Box) : \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta} \quad \frac{\Box\Gamma \Rightarrow \Box\Delta, \varphi}{\Box\Gamma, \Rightarrow \Box\Delta, \Box\varphi} \quad [\text{Globalization}].$$

(1) Since  $QL \vdash (\varphi \rightarrow \varphi)$ , if  $QL \vdash \varphi, \Gamma \Rightarrow \Delta$ , then by  $(\rightarrow)$ ,

$$QL \vdash ((\varphi \rightarrow \varphi) \rightarrow \varphi), \Gamma \Rightarrow \Delta, \quad \text{i.e.} \quad QL \vdash \Box\varphi, \Gamma \Rightarrow \Delta.$$

(2) If  $QL \vdash \Box\Gamma \Rightarrow \Box\Delta, \varphi$ , then  $QL \vdash (\varphi \rightarrow \varphi), \Box\Gamma \Rightarrow \Box\Delta, \varphi$ . Therefore,  $QL \vdash \Box\Gamma \Rightarrow \Box\Delta, \Box\varphi$ .

□

## Chapter 6

# Various Global Set Theories

Formal classical set theory ZFC is the system consisting of classical logic LK and Zermelo-Freankel axioms A1-A9 and AC.

Truth value set of LK is complete Boolean lattice, i.e. complete Boolean lattice is a counterpart of classical logic. D.Scott and R.Solovey introduced a Boolean valued universe. Thus, ZFC is described and interpreted in the Boolean valued universe.

The usual mathematics can be developed in the Boolean valued universe, whereas Boolean valued universe is constructed in ZFC.

An intuitionistic logic is represented by a complete Heyting algebra and the intuitionistic set theory is developed in a Heyting valued universe. Boolean algebra and Heyting algebra have the structure of sheaf of complete Boolean algebra  $\mathbf{2} = \{1, 0\}$ . Both of Heyting valued universe and Boolean valued universe have the structure of sheaf of  $\mathbf{2}$ -valued universe.

Classical set theory ZFC is not complete as shown in Gödel's incompleteness theorem. That is, there exists a valid formula of ZFC which is not provable in ZFC.

Von-Neumann-Bernays-Gödel set theory NBG has the notion of class, which is a collection of sets defined by a formula whose quantifiers range over sets. NBG is a conservative extension of ZFC, and can define classes such as the class of all sets and the class of all ordinals.

Global Von-Neumann-Bernays-Gödel set theory (GNBG) is the global classical logic with axioms of global set theory.

A meta-theoretical sentence “A formula  $\varphi$  is true” is expressible in GNBG. As a result, GNBG becomes complete.

## 6.1 Global set theory

Global set theory GNBG is based on global logic with primitive symbols

- Class constant  $V$ , which indicates ‘universe’
- Set constant  $\emptyset$ , which indicates ‘empty set’
- Class variables  $A, B, C, \dots, X, Y, Z, \dots$
- Set variables  $a, b, c, \dots, x, y, z, \dots$
- membership relation  $\in$  and equality  $=$
- Logical operators  $\wedge, \vee, \rightarrow$  ( or  $\Box$ ),  $\forall$  and  $\exists$ .

Atomic formulas of GNBG are of the form  $A = B$  or  $a \in A$ .

### 6.1.1 Axioms of global set theory

Axioms for global set theory are similar to those of ZFC set theory. Quantifiers of the form  $\forall x \in V$  or  $\exists x \in V$  for set variables will shorten to  $\forall x$  or  $\exists x$ .

**GA1. Universe**  $\forall X (\exists Y (X \in Y) \rightarrow \exists x \in V (x = X))$

**Empty set**  $\forall x \neg (x \in \emptyset)$

**GA2. Extensionality**  $\forall X, Y (X = Y \leftrightarrow \forall z (z \in X \leftrightarrow z \in Y))$ .

**GA3. Regularity**  $\forall x (\exists y (y \in x) \rightarrow \exists y \in x (y \cap x = \emptyset))$ .

**GA4. Infinity**  $\exists x (\exists a (a \in x) \wedge \forall y \in x \exists z (y \in z \wedge z \in x))$

**GA5. Pairing**  $\forall u, v \exists z (\forall x (x \in z \leftrightarrow (x = u \vee x = v)))$ .

The set  $z$  satisfying  $\forall x (x \in z \leftrightarrow (x = u \vee x = v))$  is denoted by  $\{u, v\}$ .

**GA6. Union**  $\forall u \exists z (\forall x (x \in z \leftrightarrow \exists y \in u (x \in y)))$ .

The set  $z$  satisfying  $\forall x (x \in z \leftrightarrow \exists y \in u (x \in y))$  is denoted by  $\bigcup u$ .

**GA7. Power set**  $\forall u \exists z (\forall x (x \in z \leftrightarrow x \subset u))$ , where

$$x \subset u \stackrel{\text{def}}{\iff} \forall y (y \in x \rightarrow y \in u).$$

The set  $z$  satisfying  $\forall x (x \in z \leftrightarrow x \subset u)$  is denoted by  $\mathcal{P}(u)$ .

**GA8. Class comprehension** For any global formula  $\varphi(x)$  containing no quantifiers over class variable, there is a class  $A$  such that

$$\forall x (x \overset{\square}{\in} A \leftrightarrow \varphi(x)).$$

The class  $A$  satisfying  $\forall x (x \overset{\square}{\in} A \leftrightarrow \varphi(x))$  is denoted by  $\{x \mid \varphi(x)\}$ .

**GA9 Collection**  $\forall u \left( \forall x (x \in u \rightarrow \exists y \varphi(x, y)) \rightarrow \exists v \forall x (x \in u \rightarrow \exists y \overset{\square}{\in} v \varphi(x, y)) \right)$ ,

where  $y \overset{\square}{\in} v \stackrel{\text{def}}{=} \square(y \in v)$ .

**GA10. Zorn**  $\text{Gl}(u) \wedge \forall v \in A (\text{Chain}(v, u) \rightarrow \bigcup v \in u) \rightarrow \exists z \in A \text{Max}(z, u)$ ,  
where

$$\begin{aligned} \text{Gl}(u) &\stackrel{\text{def}}{\iff} \forall x (x \in u \rightarrow x \overset{\square}{\in} u), \\ \text{Chain}(v, u) &\stackrel{\text{def}}{\iff} v \subset u \wedge \forall x, y (x, y \in v \rightarrow x \subset y \vee y \subset x), \\ \text{Max}(z, u) &\stackrel{\text{def}}{\iff} z \in u \wedge \forall x (x \in u \wedge z \subset x \rightarrow z = x). \end{aligned}$$

**GA11. Axiom of  $\Diamond$**   $\forall U \exists Z \forall t (t \in Z \leftrightarrow \Diamond(t \in U))$ .

The class  $Z$  satisfying  $\forall t (t \in Z \leftrightarrow \Diamond(t \in U))$  is denoted by  $\Diamond U$ .

## 6.2 Basic universe of set theory

Underlying basic universe  $V (= V^{\mathbf{2}})$  (cf. p.78) is constructed inductively:

$$\begin{aligned} V_\alpha &= \{u \mid \exists \beta < \alpha \exists \mathcal{D}u \subset V_\beta (u : \mathcal{D}u \rightarrow \mathbf{2})\}, \\ V &= \bigcup_{\alpha \in On} V_\alpha. \end{aligned}$$

Truth values of atomic formulas in the basic universe are given as

$$\begin{aligned}\llbracket u=v \rrbracket &= \bigwedge_{x \in \mathcal{D}u} (u(x) \supset \llbracket x \in v \rrbracket) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \supset \llbracket x \in u \rrbracket), \\ \llbracket u \in v \rrbracket &= \bigvee_{x \in \mathcal{D}v} (v(x) \wedge \llbracket u=x \rrbracket).\end{aligned}$$

Logical operators  $\wedge, \vee, \neg, \forall, \exists$  are represented by algebraic operators on the Boolean algebra **2**.

$$\begin{aligned}\llbracket \varphi \wedge \psi \rrbracket &= \llbracket \varphi \rrbracket \wedge \llbracket \psi \rrbracket \\ \llbracket \varphi \vee \psi \rrbracket &= \llbracket \varphi \rrbracket \vee \llbracket \psi \rrbracket \\ \llbracket \varphi \rightarrow \psi \rrbracket &= \begin{cases} 1 & \llbracket \varphi \rrbracket \leq \llbracket \psi \rrbracket \\ 0 & \text{otherwise} \end{cases} \\ \llbracket \Box \varphi \rrbracket &= \begin{cases} 1 & \llbracket \varphi \rrbracket \neq 1 \\ 0 & \text{otherwise} \end{cases} \\ \llbracket \forall x \varphi(x) \rrbracket &= \bigwedge_{x \in V} \llbracket \varphi(x) \rrbracket \\ \llbracket \exists x \varphi(x) \rrbracket &= \bigvee_{x \in V} \llbracket \varphi(x) \rrbracket\end{aligned}$$

Then every formula has truth value 1 or 0. We express mathematical objects such as numbers, relations, functions and etc. in the universe.

### 6.3 Lattice valued universe

In general, truth value set of global set theory is an complete lattice. Let  $\mathcal{L}$  be a complete lattice, where  $\mathcal{L}$  is a set. Logical operators are interpreted as algebraic operators on the complete lattice, where  $\rightarrow$  is the basic implication and  $\neg$  is the corresponding negation:

$$\neg \varphi \stackrel{\text{def}}{\iff} \varphi \rightarrow \perp.$$

$\mathcal{L}$ -valued universe  $V^{\mathcal{L}}$  is constructed in  $V$  by induction:

$$\begin{aligned}V_{\alpha}^{\mathcal{L}} &= \{u \mid \exists \beta < \alpha \exists \mathcal{D}u \subset V_{\beta}^{\mathcal{L}} (u : \mathcal{D}u \rightarrow \mathcal{L})\} \\ V^{\mathcal{L}} &= \bigcup_{\alpha \in \text{On}} V_{\alpha}^{\mathcal{L}}\end{aligned}$$

The least  $\alpha$  such that  $u \in V_\alpha^\mathcal{L}$  is called the **rank** of  $u$ . Class  $A$  of  $V^\mathcal{L}$  is a function  $\mathcal{D}A \rightarrow \mathcal{L}$ , where the domain  $\mathcal{D}A$  is a sub-class of  $V^\mathcal{L}$ .

$$A : \mathcal{D}A \rightarrow \mathcal{L}$$

For classes  $A, B$  and a set  $a$ ,  $\llbracket A=B \rrbracket$  and  $\llbracket a \in B \rrbracket$  are defined by induction on the rank of  $A, B$ .

$$\begin{aligned} \llbracket A=B \rrbracket &= \bigwedge_{x \in \mathcal{D}A} (A(x) \rightarrow \llbracket x \in B \rrbracket) \wedge \bigwedge_{x \in \mathcal{D}B} (B(x) \rightarrow \llbracket x \in A \rrbracket) \\ \llbracket a \in B \rrbracket &= \bigvee_{x \in \mathcal{D}B} (\llbracket a=x \rrbracket \wedge B(x)). \end{aligned}$$

We say an element  $p$  of  $\mathcal{L}$  is  $\Box$ -**closed** if  $p = \Box p$ , where  $\Box p = ((p \rightarrow p) \rightarrow p)$ . As an immediate consequence of the definition of atomic formulas, we have:

**LEMMA 6.3.1.**  $\llbracket A=B \rrbracket$  is  $\Box$ -closed for every classes  $A, B$ .

Hence we have

**LEMMA 6.3.2.** For every classed  $A, B$  and  $\{b_k\}_k \subset \mathcal{L}$ ,

$$\llbracket A=B \rrbracket \wedge \bigvee_k b_k = \bigvee_k \llbracket A=B \rrbracket \wedge b_k;$$

and for classes  $A$  and  $B_k$  and  $b \in \mathcal{L}$ ,

$$(\bigvee_k \llbracket A = B_k \rrbracket) \wedge b = \bigvee_k (\llbracket A = B_k \rrbracket \wedge b).$$

**LEMMA 6.3.3.** Let  $A, B$  are classes of  $V^\mathcal{L}$ . Then

- (1)  $\llbracket A=B \rrbracket = \llbracket B=A \rrbracket$
- (2)  $\llbracket A=A \rrbracket = 1$
- (3) If  $x \in \mathcal{D}A$  then  $A(x) \leq \llbracket x \in A \rrbracket$ .

*Proof.* (1) is obvious.

Let  $x \in \mathcal{D}A$ . Since  $\llbracket x=x \rrbracket = 1$  by induction hypothesis,

$$A(x) \leq \bigvee_{x' \in \mathcal{D}A} \llbracket x=x' \rrbracket \wedge A(x') \leq \llbracket x \in A \rrbracket,$$

and hence,  $\llbracket A=A \rrbracket = 1$ . □

**THEOREM 6.3.4.** *For classes  $A, B, C, a, b$  of  $V^{\mathcal{L}}$ ,*

- (1)  $\llbracket A=B \wedge B=C \rrbracket \leq \llbracket A=C \rrbracket$
- (2)  $\llbracket A=B \wedge B \in C \rrbracket \leq \llbracket B \in C \rrbracket$
- (3)  $\llbracket A=B \wedge C \in B \rrbracket \leq \llbracket C \in A \rrbracket$

*Proof.* (1) We proceed by induction. Assume that  $A, B, C$  are classes of  $V_{\alpha}^{\mathcal{L}}$ . By Theorem 3.7.5.(8),

$$\llbracket A=B \rrbracket \wedge A(x) \leq (A(x) \rightarrow \llbracket x \in B \rrbracket) \wedge A(x) \leq \llbracket x \in B \rrbracket$$

for  $x \in \mathcal{D}A$ . Since  $\llbracket A=B \wedge B=C \rrbracket$  is  $\square$ -closed,

$$\begin{aligned} \llbracket A=B \wedge B=C \rrbracket \wedge A(x) &\leq \llbracket B=C \rrbracket \wedge \bigvee_{y \in \mathcal{D}B} \llbracket x=y \rrbracket \wedge B(y) \\ &\leq \bigvee_{y \in \mathcal{D}B} (\llbracket x=y \rrbracket \wedge \llbracket B=C \rrbracket \wedge B(y)) \\ &\leq \bigvee_{y \in \mathcal{D}B} (\llbracket x=y \rrbracket \wedge \bigvee_{z \in \mathcal{D}C} \llbracket y=z \rrbracket \wedge C(z)) \\ &\leq \bigvee_{y \in \mathcal{D}B} \bigvee_{z \in \mathcal{D}C} \llbracket x=y \wedge y=z \rrbracket \wedge C(z). \end{aligned}$$

By using induction hypothesis,

$$\begin{aligned} &\leq \bigvee_{z \in \mathcal{D}C} \llbracket x=z \rrbracket \wedge C(z) \\ &\leq \llbracket x \in C \rrbracket. \end{aligned}$$

Since  $\llbracket A=B \wedge B=C \rrbracket$  is  $\square$ -closed again,

$$\llbracket A=B \wedge B=C \rrbracket \leq \bigwedge_{x \in \mathcal{D}A} (A(x) \rightarrow \llbracket x \in C \rrbracket).$$

Similarly, we have

$$\llbracket A=B \wedge B=C \rrbracket \leq \bigwedge_{z \in \mathcal{D}} (C(z) \rightarrow \llbracket z \in A \rrbracket).$$

Hence,  $\llbracket A=B \wedge B=C \rrbracket \leq \llbracket A=C \rrbracket$ .

(2) and (3) follows from (1). □

## 6.4 Lattice valued set theory

A global set theory on  $V^{\mathcal{L}}$  whose primitive formulas are of the form  $A = B$  or  $a \in B$  and logical operations are  $\wedge, \vee, \neg, \rightarrow, \forall x, \exists x$  is called **lattice valued set theory** LNBG . We extend the definition of  $\llbracket \varphi \rrbracket$  in natural way:

$$\begin{aligned}\llbracket \neg \varphi \rrbracket &= \neg \llbracket \varphi \rrbracket \\ \llbracket \varphi_1 \wedge \varphi_2 \rrbracket &= \llbracket \varphi_1 \rrbracket \wedge \llbracket \varphi_2 \rrbracket \\ \llbracket \varphi_1 \vee \varphi_2 \rrbracket &= \llbracket \varphi_1 \rrbracket \vee \llbracket \varphi_2 \rrbracket \\ \llbracket \varphi_1 \rightarrow \varphi_2 \rrbracket &= \llbracket \varphi_1 \rrbracket \rightarrow \llbracket \varphi_2 \rrbracket = \begin{cases} 1 & \llbracket \varphi_1 \rrbracket \leq \llbracket \varphi_2 \rrbracket \\ 0 & \llbracket \varphi_1 \rrbracket \not\leq \llbracket \varphi_2 \rrbracket \end{cases} \\ \llbracket \forall x \varphi(x) \rrbracket &= \bigwedge_{u \in V^{\mathcal{L}}} \llbracket \varphi(u) \rrbracket \\ \llbracket \exists x \varphi(x) \rrbracket &= \bigvee_{u \in V^{\mathcal{L}}} \llbracket \varphi(u) \rrbracket \\ \llbracket \Box \varphi \rrbracket &= \Box \llbracket \varphi \rrbracket, \text{ where } \Box \varphi \text{ denotes } (\varphi \rightarrow \varphi) \rightarrow \varphi.\end{aligned}$$

The equality axioms are valid on  $V^{\mathcal{L}}$  :

**THEOREM 6.4.1.** *For any formula  $\varphi(X)$  and classes  $A, B$  of  $V^{\mathcal{L}}$ ,*

$$\llbracket A = B \wedge \varphi(A) \rrbracket \leq \llbracket \varphi(B) \rrbracket.$$

*Proof.* If  $\varphi(X)$  is an atomic formula, then it is immediate from Theorem 6.3.3 and 6.3.4. Other cases follows from the fact that  $\llbracket A = B \rrbracket$  is  $\Box$ -closed and Theorem 5.4.2.  $\square$

**THEOREM 6.4.2.** *For any formula  $\varphi(x)$  and class  $A$  of  $V^{\mathcal{L}}$ ,*

- (1)  $\llbracket \forall x(x \in A \rightarrow \varphi(x)) \rrbracket = \bigwedge_{x \in \mathcal{D}A} \llbracket x \in A \rightarrow \varphi(x) \rrbracket$
- (2)  $\llbracket \exists x(x \in A \wedge \varphi(x)) \rrbracket = \bigvee_{x \in \mathcal{D}A} \llbracket x \in A \wedge \varphi(x) \rrbracket$

*Proof.* (1):  $\llbracket \forall x(x \in A \rightarrow \varphi(x)) \rrbracket \leq \bigwedge_{x \in \mathcal{D}A} \llbracket x \in A \rightarrow \varphi(x) \rrbracket$  is obvious. Now we show  $(\geq)$ . By using the fact that  $\llbracket x \in A \rrbracket \leq \bigvee_{x' \in \mathcal{D}A} \llbracket x = x' \rrbracket$ , we have

$$\begin{aligned}\bigwedge_{x' \in \mathcal{D}A} \llbracket x' \in A \rightarrow \varphi(x') \rrbracket \wedge \llbracket x \in A \rrbracket &= \bigwedge_{x' \in \mathcal{D}A} \llbracket x' \in A \rightarrow \varphi(x') \rrbracket \wedge \llbracket x \in u \rrbracket \wedge \bigvee_{x'' \in \mathcal{D}A} \llbracket x = x'' \rrbracket \\ &= \bigvee_{x'' \in \mathcal{D}A} \left( \bigwedge_{x' \in \mathcal{D}A} \llbracket x' \in A \rightarrow \varphi(x') \rrbracket \wedge \llbracket x \in A \rrbracket \wedge \llbracket x = x'' \rrbracket \right) \\ &\leq \llbracket \varphi(x) \rrbracket\end{aligned}$$

Since  $\bigwedge_{x \in \mathcal{D}A} \llbracket x \in A \rightarrow \varphi(x) \rrbracket$  is  $\Box$ -closed, we have

$$\bigwedge_{x \in \mathcal{D}A} \llbracket x \in A \rightarrow \varphi(x) \rrbracket \leq \llbracket \forall x (x \in A \rightarrow \varphi(x)) \rrbracket.$$

(2): By using  $\llbracket x \in A \rrbracket \leq \bigvee_{x \in \mathcal{D}A} \llbracket x = x' \rrbracket$  again,

$$\begin{aligned} \llbracket \exists x (x \in A \wedge \varphi(x)) \rrbracket &\leq \bigvee_{x \in V^{\mathcal{L}}} \bigvee_{x' \in \mathcal{D}A} (\llbracket x = x' \rrbracket \wedge \llbracket x \in A \wedge \varphi(x) \rrbracket) \\ &\leq \bigvee_{x' \in \mathcal{D}A} \llbracket x' \in A \wedge \varphi(x') \rrbracket. \quad \square \end{aligned}$$

**DEFINITION 6.4.1.** Restriction  $A \upharpoonright p$  of class  $A$  of  $V^{\mathcal{L}}$  by  $p \in \mathcal{L}$  is defined by

$$\begin{cases} \mathcal{D}(A \upharpoonright p) = \{x \upharpoonright p \mid x \in \mathcal{D}A\} \\ (A \upharpoonright p)(x \upharpoonright p) = \bigvee \{A(x') \wedge p \mid x' \in \mathcal{D}A, x \upharpoonright p = x' \upharpoonright p\} \text{ for } x \in \mathcal{D}A. \end{cases}$$

If  $A \in V_{\alpha}^{\mathcal{L}}$ , so is  $A \upharpoonright p$ , and we have

**THEOREM 6.4.3.** If  $A$  is a class of  $V^{\mathcal{L}}$ ,  $x \in V^{\mathcal{L}}$ ,  $p, q \in \mathcal{L}$ , and  $p$  is  $\Box$ -closed (i.e.,  $p = \Box p$ ), then

- (1)  $p \leq \llbracket A = A \upharpoonright p \rrbracket$
- (2)  $\llbracket x \in A \upharpoonright p \rrbracket = \llbracket x \in A \rrbracket \wedge p$
- (3)  $(A \upharpoonright q) \upharpoonright p = A \upharpoonright (p \wedge q)$ .

*Proof.* We proceed by induction on the rank of  $\mathcal{D}A$ ,

- (1) : For  $x \in \mathcal{D}A$ ,

$$\begin{aligned} p \wedge A(x) &\leq (A \upharpoonright p)(x \upharpoonright p) \wedge \llbracket x = x \upharpoonright p \rrbracket \leq \llbracket x \in A \upharpoonright p \rrbracket \\ (A \upharpoonright p)(x \upharpoonright p) &= \bigvee_{x' \in \mathcal{D}A, x \upharpoonright p = x' \upharpoonright p} A(x') \wedge p \wedge \llbracket x = x' = x \upharpoonright p \rrbracket \leq \llbracket x \upharpoonright p \in A \rrbracket. \end{aligned}$$

Therefore,  $p \leq \llbracket A = A \upharpoonright p \rrbracket$ .

(2) : By (1) and Theorem 6.4.1,

$$\llbracket x \in A \rrbracket \wedge p \leq \llbracket x \in A \upharpoonright p \rrbracket.$$

( $\leq$ ) follows from the fact that  $x'' \upharpoonright p = x' \upharpoonright p$  implies  $p \leq \llbracket x'' = x' \rrbracket$  :

$$\begin{aligned} \llbracket x \in A \upharpoonright p \rrbracket &= \bigvee_{x' \in \mathcal{D}A} \llbracket x = x' \upharpoonright p \rrbracket \wedge \bigvee_{x'' \in \mathcal{D}A, x'' \upharpoonright p = x' \upharpoonright p} A(x'') \wedge p \\ &\leq \llbracket x \in A \rrbracket \wedge p \end{aligned}$$

(3) :  $\mathcal{D}((A \upharpoonright q) \upharpoonright p) = \mathcal{D}(A \upharpoonright (q \wedge p))$ , by the induction hypothesis, and

$$((A \upharpoonright q) \upharpoonright p)((x \upharpoonright q) \upharpoonright p) = (u \upharpoonright (q \wedge p))(x \upharpoonright (q \wedge p))$$

by using the fact :  $(\bigvee_{x'} (A(x') \wedge q)) \wedge p = \bigvee_{x'} (A(x') \wedge q \wedge p)$ .

□

**DEFINITION 6.4.2.** If  $\varphi$  is valid in  $V^{\mathcal{L}}$ , i.e.  $\llbracket \varphi \rrbracket = 1$  in  $V^{\mathcal{L}}$ , then we write  $V^{\mathcal{L}} \models \varphi$ .

$$V^{\mathcal{L}} \models \varphi \stackrel{\text{def}}{\iff} (\varphi \text{ is valid in } V^{\mathcal{L}})$$

**THEOREM 6.4.4.** Axioms of global set theory are valid in the universe  $V^{\mathcal{L}}$ .

*Proof.* We prove  $V^{\mathcal{L}} \models \varphi$  for each axiom  $\varphi$  of global set theory.

**Axiom of extensionality:**  $\forall x(x \in A \leftrightarrow x \in B) \rightarrow A = B$ .

*Proof.* By Theorem 6.4.2.

□

**Axiom of regularity:**  $\forall x(\exists y(y \in x) \rightarrow \exists y \in x(y \cap x = \emptyset))$ .

*Proof.* Let  $y$  be an element of minimum rank of  $\mathcal{D}x$ .

□

**Axiom of pair:**  $\forall u, v \exists z \forall x(x \in z \leftrightarrow x = u \vee x = v)$ .

*Proof.* For  $u, v \in V^{\mathcal{L}}$  define  $z$  by

$$\begin{cases} \mathcal{D}z = \{u, v\} \\ z(t) = 1 \quad \text{for } t \in \mathcal{D}z \end{cases}$$

Then  $\llbracket x \in z \rrbracket = \bigvee_{t \in \mathcal{D}z} \llbracket x = t \rrbracket \wedge z(t) = \llbracket x = u \rrbracket \vee \llbracket x = v \rrbracket$ .

Therefore,  $\llbracket \forall x(x \in z \leftrightarrow x = u \vee x = v) \rrbracket = 1$ . □

**Axiom of union:**  $\forall u \exists v \forall x(x \in v \leftrightarrow \exists y(y \in u \wedge x \in y))$ .

*Proof.* For  $u \in V^{\mathcal{L}}$  defined  $v$  by

$$\begin{cases} \mathcal{D}v = \bigcup_{y \in \mathcal{D}u} \mathcal{D}y \\ v(x) = \llbracket \exists y(y \in u \wedge x \in y) \rrbracket. \end{cases}$$

Then, by Theorem 6.4.2,

$$\begin{aligned} \llbracket \exists y(y \in u \wedge x \in y) \rrbracket &= \bigvee_{y \in \mathcal{D}u} \llbracket y \in u \rrbracket \wedge \llbracket x \in y \rrbracket \\ &= \bigvee_{y \in \mathcal{D}u} \llbracket y \in u \rrbracket \wedge \llbracket x \in y \rrbracket \wedge \bigvee_{x' \in \mathcal{D}y} \llbracket x = x' \rrbracket \\ &= \bigvee_{y \in \mathcal{D}u, x' \in \mathcal{D}y} \llbracket x = x' \rrbracket \wedge \llbracket x' \in y \rrbracket \wedge \llbracket y \in u \rrbracket \\ &= \llbracket x \in v \rrbracket \end{aligned}$$

□

**DEFINITION 6.4.3.** For each set  $x \in V$  we define  $\check{x} \in V^{\mathcal{L}}$  by

$$\begin{aligned} \mathcal{D}\check{x} &= \{\check{t} \mid t \in x\} \\ \check{x}(\check{t}) &= 1. \end{aligned}$$

$\check{x}$  is called the **check set associated with**  $x$ . For check sets  $\check{x}, \check{y}$ , we have

$$\llbracket \check{x} = \check{y} \rrbracket = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} ; \quad \llbracket \check{x} \in \check{y} \rrbracket = \begin{cases} 1 & \text{if } x \in y \\ 0 & \text{if } x \notin y. \end{cases} \quad (6.4.1)$$

**DEFINITION 6.4.4.** “ $u$  is a check set”, in symbols  $\text{ck}(u)$ , is defined by

$$\text{ck}(u) \stackrel{\text{def}}{\iff} \forall t(t \in u \rightarrow t \overset{\square}{\in} u \wedge \text{ck}(t)),$$

where  $t \overset{\square}{\in} u \stackrel{\text{def}}{=} \square(t \in u)$ . Then  $\llbracket \text{ck}(\check{x}) \rrbracket = 1$  for all  $x$ .

**Axiom of infinity:**  $\exists u (\exists x(x \in u) \wedge \forall x(x \in u \rightarrow \exists y \in u(x \in y)))$ .

*Proof.*  $\check{\omega}$  associated with the set  $\omega$  of all natural numbers satisfies

$$\llbracket \exists x(x \in \check{\omega}) \wedge \forall x(x \in \check{\omega} \rightarrow \exists y \in \check{\omega}(x \in y)) \rrbracket = 1.$$

□

**Axiom of power set:**  $\forall u \exists v \forall x(x \in v \leftrightarrow x \subset u)$ , where

$$x \subset u \stackrel{\text{def}}{\iff} \forall t(t \in x \rightarrow t \in u).$$

*Proof.* Let  $u \in V_{\alpha}^{\mathcal{L}}$ . For every  $x \in V^{\mathcal{L}}$ , define  $x^*$  by

$$\begin{cases} \mathcal{D}x^* = \mathcal{D}u \\ x^*(t) = \llbracket x \subset u \wedge t \in x \rrbracket. \end{cases}$$

Since

$$\llbracket x \subset u \wedge t \in x \rrbracket \leq \llbracket t \in u \rrbracket \leq \bigvee_{t' \in \mathcal{D}u} \llbracket t = t' \rrbracket,$$

we have

$$\begin{aligned} \llbracket x \subset u \wedge t \in x \rrbracket &\leq \bigvee_{t' \in \mathcal{D}u} \llbracket t = t' \wedge x \subset u \wedge t' \in x \rrbracket \\ &\leq \llbracket t \in x^* \rrbracket. \end{aligned}$$

It follows that for every  $x \in V^{\mathcal{L}}$  there exists  $x^* \in V_{\alpha+1}^{\mathcal{L}}$  such that

$\llbracket x \subset u \rrbracket \leq \llbracket x = x^* \rrbracket$ . Now we define  $v$  by

$$\begin{cases} \mathcal{D}v = \{x \in V_{\alpha+1}^{\mathcal{L}} \mid \mathcal{D}x = \mathcal{D}u\} \\ v(x) = \llbracket x \subset u \rrbracket. \end{cases}$$

Then

$$\llbracket \forall x(x \in v \leftrightarrow x \subset u) \rrbracket = 1.$$

□

**Axiom of class comprehension:** For a first order formula  $\varphi(a)$  with free variable  $a$  and class parameters,  $\exists X (x \in X \leftrightarrow \varphi(x))$ .

*Proof.* Define  $X$  by

$$X(x) = \llbracket \varphi(x) \rrbracket$$

□

We denote  $\Box(a \in b)$  by  $a \overset{\Box}{\in} b$ .

**Axiom of collection:**  $\forall u \left( \forall x (x \in u \rightarrow \exists y \varphi(x, y)) \rightarrow \exists v \forall x (x \in u \rightarrow \exists y \overset{\Box}{\in} v \varphi(x, y)) \right)$ .

*Proof.* Let

$$p = \llbracket \forall x (x \in u \rightarrow \exists y \varphi(x, y)) \rrbracket = \bigwedge_{x \in \mathcal{D}u} (\llbracket x \in u \rrbracket \rightarrow \bigvee_y \llbracket \varphi(x, y) \rrbracket).$$

It suffices to show that there exists  $v$  such that

$$p \leq \llbracket \forall x (x \in u \rightarrow \exists y \overset{\Box}{\in} v \varphi(x, y)) \rrbracket.$$

Since  $\mathcal{L}$  is a set, for each  $x \in \mathcal{D}u$  there exists an ordinal  $\alpha(x)$  such that

$$p \wedge \llbracket x \in u \rrbracket \leq \bigvee_{y \in V_{\alpha(x)}^{\mathcal{L}}} \llbracket \varphi(x, y) \rrbracket.$$

Hence, by using the axiom of collection externally, there exists an ordinal  $\alpha$  such that

$$p \wedge \llbracket x \in u \rrbracket \leq \bigvee_{y \in V_{\alpha}^{\mathcal{L}}} \llbracket \varphi(x, y) \rrbracket \quad \text{for all } x \in \mathcal{D}u.$$

Now we defined  $v$  by

$$\begin{cases} \mathcal{D}v = V_{\alpha}^{\mathcal{L}} \\ v(y) = 1 \end{cases}$$

Then

$$p \wedge \llbracket x \in u \rrbracket \leq \bigvee_{y \in \mathcal{D}v} \llbracket y \overset{\Box}{\in} v \wedge \varphi(x, y) \rrbracket = \llbracket \exists y \overset{\Box}{\in} v \varphi(x, y) \rrbracket \quad \text{for all } x \in \mathcal{D}u.$$

Since  $p = \Box p$ , we have

$$p \leq \llbracket \forall x (x \in u \rightarrow \exists y \overset{\Box}{\in} v \varphi(x, y)) \rrbracket.$$

□

**Axiom of  $\in$ -induction:**  $\forall x (\forall y (y \in x \rightarrow \varphi(y)) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x)$ .

*Proof.* Let  $p = \llbracket \forall x (\forall y (y \in x \rightarrow \varphi(y)) \rightarrow \varphi(x)) \rrbracket$ . We prove  $p \leq \llbracket \forall x \varphi(x) \rrbracket = \bigwedge_{x \in V^{\mathcal{L}}} \llbracket \varphi(x) \rrbracket$  by induction on the rank of  $x$ . Let  $x \in V_{\alpha}^{\mathcal{L}}$ . Since  $p \leq \llbracket \varphi(y) \rrbracket$  for all  $y \in \mathcal{D}x \subset V_{<\alpha}^{\mathcal{L}}$  by induction hypothesis,

$$p \wedge \llbracket y \in x \rrbracket \leq \llbracket \varphi(y) \rrbracket \text{ for all } y \in \mathcal{D}x.$$

Hence, by using  $p = \Box p$ , we have

$$p \leq \llbracket \forall y (y \in x \rightarrow \varphi(y)) \rrbracket.$$

It follows that  $p \leq \llbracket \forall x \varphi(x) \rrbracket$ .  $\square$

**Zorn's Lemma :**  $\text{Gl}(u) \wedge \forall v [\text{Chain}(v, u) \rightarrow \bigcup v \in u] \rightarrow \exists z \text{Max}(z, u)$ , where

$$\begin{aligned} \text{Gl}(u) &\stackrel{\text{def}}{\iff} \forall x (x \in u \rightarrow x \overset{\Box}{\in} u), \\ \text{Chain}(v, u) &\stackrel{\text{def}}{\iff} v \subset u \wedge \forall x, y (x, y \in v \rightarrow x \subset y \vee y \subset x), \\ \text{Max}(z, u) &\stackrel{\text{def}}{\iff} z \in u \wedge \forall x (x \in u \wedge z \subset x \rightarrow z = x). \end{aligned}$$

*Proof.* For  $u \in V_{\alpha}^{\mathcal{L}}$ , let

$$p = \llbracket \text{Gl}(u) \wedge \forall v (\text{Chain}(v, u) \rightarrow \bigcup v \in u) \rrbracket,$$

and let  $U$  be a maximal subset of  $V_{\alpha}^{\mathcal{L}}$  such that

$$\forall x, y \in U (\llbracket x \in u \wedge \exists t (t \in x) \wedge y \in u \wedge \exists t (t \in y) \rrbracket \wedge p \leq \llbracket x \subset y \vee y \subset x \rrbracket).$$

$U$  is not empty. Define  $v$  by

$$\begin{cases} \mathcal{D}v = U \\ v(x) = p \wedge \llbracket x \in u \wedge \exists t (t \in x) \rrbracket. \end{cases}$$

Now we prove that  $p \leq \llbracket \text{Max}(\bigcup v, u) \rrbracket$ . Since  $p = \Box p$  and  $p \wedge v(x) \leq \llbracket x \in u \rrbracket$  for all  $x \in \mathcal{D}v$ , we have  $p \leq \llbracket v \subset u \rrbracket$ . Hence, by the definition of  $v$ ,  $p \leq \llbracket \text{Chain}(v, u) \rrbracket$ . Therefore,  $p \leq \llbracket \bigcup v \in u \rrbracket$ . Now it suffices to show that

$$p \wedge \llbracket x \in u \wedge \bigcup v \subset x \rrbracket \leq \llbracket x \subset \bigcup v \rrbracket \text{ for } x \in \mathcal{D}u.$$

Let  $x \in \mathcal{D}u$  and  $r = p \wedge \llbracket x \in u \wedge \bigcup v \subset x \rrbracket$ . Then  $r$  is  $\Box$ -closed, and we have  $r \leq \llbracket x = x \restriction r \rrbracket$  by Theorem 6.4.3. Hence  $x \restriction r \in U$ . In fact, for each  $y \in U$ , we have

$$\begin{aligned} \llbracket y \in u \wedge \exists t(t \in y) \wedge (x \restriction r) \in u \wedge \exists t(t \in x \restriction r) \rrbracket \wedge p &\leq \llbracket y \in v \rrbracket \wedge r \\ &\leq \llbracket y \subset \bigcup v \subset x \rrbracket \wedge \llbracket x = x \restriction r \rrbracket \\ &\leq \llbracket y \subset x \restriction r \rrbracket \\ &\leq \llbracket y \subset x \restriction r \vee x \restriction r \subset y \rrbracket. \end{aligned}$$

It follows that

$$\begin{aligned} r \wedge x(t) &\leq \llbracket x = x \restriction r \wedge x \in u \wedge t \in x \rrbracket \wedge p \\ &\leq \llbracket x = x \restriction r \wedge x \restriction r \in u \wedge \exists t(t \in x \restriction r) \rrbracket \wedge p \\ &\leq \llbracket x = x \restriction r \rrbracket \wedge v(x \restriction r) \\ &\leq \llbracket x \in v \rrbracket \leq \llbracket x \subset \bigcup v \rrbracket \end{aligned}$$

Therefore,  $r \leq \llbracket x \subset \bigcup v \rrbracket$ . □

**DEFINITION 6.4.5.**  $\Diamond$  is the logical operation defined by  $\Diamond\varphi \stackrel{\text{def}}{\iff} \neg\Box\neg\varphi$ .

**Axiom of  $\Diamond$ :**  $\forall u\exists v\forall x(x \in v \leftrightarrow \Diamond(x \in u))$ .

*Proof.* For a given  $u \in V^{\mathcal{L}}$ , defined  $v$  by

$$\begin{cases} \mathcal{D}v = \mathcal{D}u \\ v(x) = \llbracket \Diamond(x \in u) \rrbracket. \end{cases}$$

By using Theorem 3.7.6,

$$\begin{aligned} \llbracket \Diamond(x \in u) \rrbracket &= \Diamond \bigvee_{x' \in \mathcal{D}u} \llbracket x = x' \rrbracket \wedge u(x') \\ &\leq \bigvee_{x' \in \mathcal{D}u} \llbracket x = x' \rrbracket \wedge \llbracket \Diamond(x' \in u) \rrbracket = \llbracket x \in v \rrbracket. \end{aligned}$$

Hence  $\llbracket \forall x(x \in v \leftrightarrow \Diamond(x \in u)) \rrbracket = 1$ . □

□

A set theory on  $V^{\mathcal{L}}$  for complete lattice  $\mathcal{L}$  is formulated as **lattice valued set theory** LNBG, which is based on the lattice valued logic.

### 6.4.1 Well-Founded Relations in LNBG

Any formula with two free set variables determines a binary relation. For a binary relation  $A(x, y)$ , we use the following abbreviations:

$$\begin{aligned} x \in \text{Dom } A &\stackrel{\text{def}}{\iff} \exists y A(x, y), & x \in \text{Rge } A &\stackrel{\text{def}}{\iff} \exists y A(y, x), \\ x \in \text{Fld } A &\stackrel{\text{def}}{\iff} \exists y (A(x, y) \vee A(y, x)). \end{aligned}$$

A binary relation  $\prec$  is said to be **well-founded** if the following conditions are satisfied:

$$\mathbf{WF1} \quad \forall x, y \neg(x \prec y \wedge y \prec x)$$

$$\mathbf{WF2} \quad \forall x [x \in \text{Fld}(\prec) \wedge \forall y (y \prec x \rightarrow \varphi(y)) \rightarrow \varphi(x)] \rightarrow \forall x (x \in \text{Fld}(\prec) \rightarrow \varphi(x))$$

$$\mathbf{WF3} \quad \forall x \exists y \forall z (z \prec x \rightarrow z \in y)$$

In view of the axiom GA9 ( $\in$ -induction), it is clear that the relation  $\in$  is itself a well-founded relation, and so is  $\overset{\square}{\in}$ .

Singleton  $\{x\}$  and ordered pair  $\langle x, y \rangle$  are defined as usual:

$$\{x\} \stackrel{\text{def}}{=} \{x, x\}, \quad \langle x, y \rangle \stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\}$$

so that  $x \in \{y\} \leftrightarrow x = y$  and  $\langle x, y \rangle = \langle x', y' \rangle \leftrightarrow x = x' \wedge y = y'$  hold.

We say a binary relation  $F(x, y)$  is **global**, if  $\forall x, y (F(x, y) \rightarrow \Box F(x, y))$ ; and a global relation  $F(x, y)$  is **functional**, if

$$\forall x, y, y' (F(x, y) \wedge F(x, y') \rightarrow y = y').$$

For a global functional relation  $F$ , we write  $F(x) = y$  instead for  $F(x, y)$ . If  $F$  is a global functional relation and  $\prec$  is a well-founded relation, then  $\{\langle x, y \rangle \mid F(x, y) \wedge \Diamond(x \prec u)\}$  is denoted by  $F_{\prec u}$ , for each set  $u \in \text{Fld}(\prec)$ .  $F_{\prec u}$  is a set by WF3, GA11( $\Diamond$ ) and GA9(Collection).

The following theorem can be proved in the usual way, by using the fact that

$$(y \prec x \rightarrow \Box \varphi(y)) \iff (\Diamond(y \prec x) \rightarrow \Box \varphi(y)).$$

**THEOREM 6.4.5** (Recursion Principle). *Let  $\prec$  be a well founded relation and  $H$  be a global functional relation such that  $\forall x \exists y H(x, y)$ . Then there exists a unique global functional relation  $F$  such that*

$$\text{Dom} F = \text{Fld}(\prec) \wedge \forall x (x \in \text{Fld}(\prec) \rightarrow (F(x) = H(F_{\prec x}))).$$

**DEFINITION 6.4.6.** *We define the formula  $\text{Ord}(\alpha)$  (“ $\alpha$  is an ordinal”) in LN BG as follows:*

$$\begin{aligned} \text{Tr}(\alpha) &\stackrel{\text{def}}{\iff} \forall \beta, \gamma (\beta \in \alpha \wedge \gamma \in \beta \rightarrow \gamma \in \alpha), \\ \text{Ord}(\alpha) &\stackrel{\text{def}}{\iff} \text{Gl}(\alpha) \wedge \text{Tr}(\alpha) \wedge \forall \beta (\beta \in \alpha \rightarrow \text{Gl}(\beta) \wedge \text{Tr}(\beta)), \end{aligned}$$

where  $\text{Gl}(\alpha) \stackrel{\text{def}}{\iff} \forall \beta (\beta \in \alpha \rightarrow \beta \sqsubset \alpha)$ .

As an immediate consequence of the above definition, we have:

**LEMMA 6.4.6.**

- (1)  $\text{Ord}(\alpha) \wedge \beta \in \alpha \rightarrow \text{Ord}(\beta)$
- (2)  $\text{Gl}(X) \wedge \forall x (x \in X \rightarrow \text{Ord}(x)) \rightarrow \text{Ord}(\bigcup X)$

**DEFINITION 6.4.7.** *A global well founded relation  $\prec$  is called a **well-ordering** on a set  $u$  if*

$$(\text{Fld}(\prec) = u) \wedge (\prec \text{ is transitive}) \wedge (\prec \text{ is extensional}),$$

where

$$\begin{aligned} \prec \text{ is transitive} &\stackrel{\text{def}}{\iff} \forall x, y, z [(x \prec y) \wedge (y \prec z) \rightarrow (x \prec z)] \\ \prec \text{ is extensional} &\stackrel{\text{def}}{\iff} \forall x, y [x, y \in u \wedge \forall z (z \prec x \leftrightarrow z \prec y) \rightarrow x = y]. \end{aligned}$$

**THEOREM 6.4.7.** *Every global set can be well-ordered, i.e. for every global set  $u$ , there exists a global well-ordering relation  $\prec$  on  $u$ .*

*Proof.* Suppose  $\text{Gl}(u)$ , and let

$$P \stackrel{\text{def}}{=} \{\langle v, w \rangle \mid \text{Gl}(v) \wedge \text{Gl}(w) \wedge v \subset u \wedge (w \text{ is a well-ordering on } v)\},$$

and let  $\langle v, w \rangle \prec \langle v', w' \rangle$  mean that  $w = w' \upharpoonright v$  and  $v$  is an initial  $w'$ -section of  $v'$ , i.e.

$$\langle v, w \rangle \prec \langle v', w' \rangle \stackrel{\text{def}}{\iff} (v \subset v') \wedge (w = w' \cap (v \times v)) \wedge (v \times (v' - v) \subset w').$$

If  $\langle v, w \rangle \in P$ , since  $\text{Gl}(v) \wedge \neg \neg(y \in v) \implies y \in v$ , we have

$$\langle v, w \rangle \prec \langle v', w' \rangle \wedge x \in v \wedge \langle y, x \rangle \in w' \implies y \in v.$$

Let

$$\mathcal{I} \stackrel{\text{def}}{=} \{I \subset P \mid \forall p, q (p, q \in I \rightarrow p \prec q \vee p = q \vee q \prec p) \wedge \forall p, q (p \in I \wedge q \prec p \rightarrow q \in I)\}.$$

Then

$$(\mathcal{I}' \subset \mathcal{I}) \wedge \forall I, I' (I, I' \in \mathcal{I}' \rightarrow I \subset I' \vee I' \subset I) \implies \bigcup \mathcal{I}' \in \mathcal{I}.$$

By using GA10, there exists a maximal  $I_0 \in \mathcal{I}$ . Let

$$v_0 = \bigcup \{v \mid \langle v, w \rangle \overset{\square}{\in} I_0\}, \quad w_0 = \bigcup \{w \mid \langle v, w \rangle \overset{\square}{\in} I_0\}.$$

Then  $\langle v_0, w_0 \rangle \in P$ . By maximality of  $I_0$  we have  $\forall x \neg(x \in u - v_0)$ .  $\forall x (x \in u \rightarrow x \in v_0 \vee \neg(x \in v_0))$ . It follows that  $u = v_0$ .  $\square$

**THEOREM 6.4.8.** *If  $u$  is a global set and  $\prec$  is a global well-ordering relation on  $u$ , then  $\langle u, \prec \rangle$  is isomorphic to an ordinal  $\langle \alpha, \in \rangle$ , i.e. there exists  $\rho$  such that*

$$(\rho: u \rightarrow \alpha) \wedge \rho(u) = \alpha \wedge \\ \forall x, y [x, y \in u \rightarrow (x \prec y \leftrightarrow \rho(x) \in \rho(y)) \wedge (x = y \leftrightarrow \rho(x) = \rho(y))].$$

*Proof.* We define by recursion in  $\prec$

$$\rho(x) = \bigcup \{\rho(y) + 1 \mid y \prec x\}.$$

It is easy to see by WF2 ( $\prec$ -induction) that  $\forall x (x \in u \rightarrow \text{Ord}(\rho(x)))$ , and

$$\forall x [x \in u \rightarrow \forall t (t \in \rho(x) \rightarrow \exists y \prec x (t = \rho(y)))].$$

Set  $\alpha = \{\rho(x) \mid x \in u\}$ . Then  $\text{Ord}(\alpha)$ , and  $\langle u, \prec \rangle$  is isomorphic to  $(\alpha, \in)$ .  $\square$

We call  $\rho(x)$  the **rank of**  $x$ .

### 6.4.2 Check sets

We define the notion of check set in LNBG, by  $\overset{\square}{\in}$ -recursion:

$$\text{ck}(x) \stackrel{\text{def}}{\iff} \forall t \left( t \in x \leftrightarrow t \overset{\square}{\in} x \wedge \text{ck}(t) \right).$$

That is, set

$$H(u, v) \stackrel{\text{def}}{\iff} v = \{t \mid \langle t, t \rangle \in u\}.$$

$H$  is a global functional relation such that  $\forall u \exists v H(u, v)$ . Let  $\prec$  be  $\overset{\square}{\in}$ .  $\prec$  is a wf relation. Since  $\forall x (x \in \text{Fld}(\prec))$ , there exists a unique global functional relation  $C(x, y)$  such that

$$\forall x [x \in \text{Dom}(C) \wedge C(x) = H(C_{\prec x})],$$

by recursion principle. If a set  $u$  satisfies  $C(u, u)$  then we say  $u$  is a **check set** and write  $\text{ck}(u)$ . i.e.

$$\text{ck}(x) \stackrel{\text{def}}{\iff} x = C(x).$$

The class of check sets will be denoted by  $W$ , i.e.

$$x \in W \stackrel{\text{def}}{\iff} \text{ck}(x).$$

**THEOREM 6.4.9.** *The followings are provable in LNBG.*

- (1)  $y \in C(x) \leftrightarrow (y \overset{\square}{\in} x) \wedge \text{ck}(y)$
- (2)  $\text{ck}(x) \leftrightarrow \forall t [t \in x \leftrightarrow (t \overset{\square}{\in} x \wedge \text{ck}(t))]$
- (3)  $C(x) = CC(x)$

*Proof.* (1) and (2) are immediate results of definition of  $C$ .

$$\begin{aligned} y \in CC(x) &\iff y \overset{\square}{\in} C(x) \wedge \text{ck}(y) \\ &\iff y \overset{\square}{\in} x \wedge \text{ck}(y) \\ &\iff y \in C(x) \end{aligned} \tag{3}$$

□

### 6.4.3 The model $W$ of ZFC in LNBG

An interpretation of ZFC in LNBG is obtained by relativizing the range of quantifiers to the class  $W$  of check sets. Namely “the class  $W$  of check sets is a model of ZFC ” is provable in LNBG.

We denote quantifiers relativized on check sets by  $\forall^W, \exists^W$ , i.e.

$$\forall^W x \varphi(x) \stackrel{\text{def}}{\iff} \forall x (\text{ck}(x) \rightarrow \varphi(x))$$

$$\exists^W x \varphi(x) \stackrel{\text{def}}{\iff} \exists x (\text{ck}(x) \wedge \varphi(x)).$$

For a formula  $\varphi$  of LNBG,  $\varphi^W$  is the formula obtained from  $\varphi$  by replacing all quantifiers  $\forall x, \exists x$ , by  $\forall^W x, \exists^W x$ , respectively.

**THEOREM 6.4.10.** *The following (1)–(9) are provable in LNBG, for any formula  $\varphi$ .*

$$(1) \quad \forall^W x, y (x \in y \rightarrow x \overset{\square}{\in} y)$$

$$(2) \quad \forall^W x_1 \cdots x_n \left( \varphi^W(x_1, \dots, x_n) \rightarrow \square \varphi^W(x_1, \dots, x_n) \right)$$

$$(3) \quad \forall^W x \left( \forall^W y (y \in x \rightarrow \varphi^W(y)) \rightarrow \varphi^W(x) \right) \rightarrow \forall^W x \varphi^W(x)$$

$$(4) \quad \forall \alpha \left( \text{Ord}(\alpha) \leftrightarrow \text{ck}(\alpha) \wedge \text{Ord}^W(\alpha) \right)$$

$$(5) \quad \text{ck}(\emptyset), \text{ where } \emptyset \text{ is the empty set.}$$

$$(6) \quad \forall^W x, y \left( \text{ck}(\{x, y\}) \wedge \text{ck}(\bigcup x) \wedge \text{ck}(\{z \in x \mid \square \varphi(z)\}) \right)$$

$$(7)$$

$$\begin{aligned} \text{Suc}(y) &\stackrel{\text{def}}{\iff} (y = \emptyset \vee \exists z (y = z + 1)), \text{ where } z + 1 = z \cup \{z\}, \\ \text{HSuc}(y) &\stackrel{\text{def}}{\iff} (\text{Suc}(y) \wedge \forall z (z \in y \rightarrow \text{Suc}(z))), \quad \text{and} \\ \omega &\stackrel{\text{def}}{=} \{y : \text{HSuc}(y)\}. \end{aligned}$$

$\omega$  is the set of natural numbers.  $\text{Ord}(\omega) \wedge \forall^W n \in \omega (n = \emptyset \vee \exists^W m \in n (n = m + 1))$ .

(8) If  $u$  is a global set, then there exists an ordinal  $\alpha \in \text{On}$  with a bijection  $\rho: u \rightarrow \alpha$ , where  $\alpha \in \text{On} \stackrel{\text{def}}{\iff} \text{Ord}(\alpha)$ , i.e.

$$\exists^W \alpha \in \text{On} \exists \rho \left( (\rho: u \rightarrow \alpha) \wedge (\rho(u) = \alpha) \wedge \forall x, y (x, y \in u \wedge \rho(x) = \rho(y) \rightarrow x = y) \right).$$

*Proof.* (1): It follows from

$$\text{ck}(x) \wedge \text{ck}(y) \wedge x \in y \iff \exists t \left( \text{ck}(x) \wedge \text{ck}(y) \wedge \text{ck}(t) \wedge x = y \wedge t \overset{\square}{\in} y \right).$$

(2): By induction on complexity of  $\varphi$ . If  $\varphi$  has no logical symbol, then  $\varphi$  is of the form  $x = y$  or  $x \in y$ , and hence  $\varphi \rightarrow \Box \varphi$  by (1). Now we prove only the case that  $\varphi$  is of the form  $\exists x \psi(x, x_1, \dots, x_n)$ , since the other cases are similar. Let  $\text{ck}(x_1) \wedge \dots \wedge \text{ck}(x_n)$ .

$$\psi^W(x, x_1, \dots, x_n) \wedge \text{ck}(x) \implies \Box (\text{ck}(x) \wedge \psi^W(x, x_1, \dots, x_n)),$$

by using induction hypothesis. Hence by Theorem 1,

$$\exists^W x \psi^W(x, x_1, \dots, x_n) \implies \Box \exists^W x \psi^W(x, x_1, \dots, x_n).$$

(3): Let  $\psi(x)$  be the formula  $\text{ck}(x) \rightarrow \varphi^W(x)$ . Then, using  $\in$ -induction, we have

$$\begin{aligned} \forall^W x \left( \forall^W y (y \in x \rightarrow \varphi^W(y)) \rightarrow \varphi^W(x) \right) &\implies \forall x \left( \forall y (y \in x \rightarrow \psi(y)) \rightarrow \psi(x) \right) \\ &\implies \forall x \psi(x). \end{aligned}$$

(4): By  $\in$ -induction.

(5):  $\text{ck}(\emptyset)$  follows from:

$$\begin{aligned} x \in \emptyset &\implies \neg(x = x) \\ &\implies x \overset{\square}{\in} \emptyset \wedge \text{ck}(x). \end{aligned}$$

(6)  $\text{ck}(\{x, y\})$ : Assume  $\text{ck}(x) \wedge \text{ck}(y)$ . Then we have

$$\begin{aligned} z \in \{x, y\} &\iff (z = x \vee z = y) \\ &\implies \text{ck}(z) \wedge z \overset{\square}{\in} \{x, y\}. \end{aligned}$$

$$\begin{aligned}
 z \in \bigcup x &\implies \exists t \in x (z \in t) \\
 &\implies \exists t \left( \text{ck}(t) \wedge t \overset{\square}{\in} x \wedge z \in t \right) \\
 &\implies \text{ck}(z) \wedge z \overset{\square}{\in} \bigcup x.
 \end{aligned}$$

$$\begin{aligned}
 t \in \{z \in x \mid \Box \varphi(z)\} &\implies \text{ck}(t) \wedge t \overset{\square}{\in} x \wedge \Box \varphi(t) \\
 &\implies \text{ck}(t) \wedge t \overset{\square}{\in} \{z \in x \mid \Box \varphi(t)\}.
 \end{aligned}$$

(7):  $\omega$  is a set by GA6 (Infinity). Let  $\psi(x)$  be the formula

$$x \in \omega \rightarrow \text{ck}(x) \wedge x \overset{\square}{\in} \omega.$$

Now we prove  $\forall y (y \in x \rightarrow \psi(y)) \rightarrow \psi(x)$ : We have  $x \in \omega \implies x = \emptyset \vee \exists z (x = z + 1)$ ,  $x = \emptyset \rightarrow \psi(x)$  and

$$\begin{aligned}
 \forall y (y \in x \rightarrow \psi(y)) \wedge x \in \omega \wedge x = z + 1 &\implies z \in x \wedge \text{ck}(z) \wedge z \overset{\square}{\in} \omega \\
 &\implies \text{ck}(z + 1) \wedge (z + 1) \overset{\square}{\in} \omega \\
 &\implies \text{ck}(x) \wedge x \overset{\square}{\in} \omega.
 \end{aligned}$$

Hence,  $\text{ck}(\omega)$ .

It is easy to see  $\forall y (y \in \omega \rightarrow \text{Tr}(y) \wedge (y \subset \omega))$ , by  $\in$ -induction, where  $\text{Tr}(y) \overset{\text{def}}{\iff} \forall s, t (s \in y \wedge t \in s \rightarrow t \in y)$ . Hence  $\text{Tr}(\omega) \wedge \forall y (y \in \omega \rightarrow \text{Tr}(y))$ .  $\text{Ord}(\omega)$  by  $\in$ -induction. It is obvious that

$$\forall^W n \in \omega (n = \emptyset \vee \exists^W m \in n (n = m + 1)).$$

(8): By Theorem 6.4.7, there exists a global well-ordering relation  $\prec$  on  $u$ . Define  $\rho(x) = \bigcup \{\rho(y) + 1 \mid y \prec x\}$ . By Theorem 6.4.8,  $\rho$  is an isomorphism between  $(u, \prec)$  and  $(\alpha, \in)$ , where  $\alpha = \{\rho(x) \mid x \in u\}$ .  $\square$

**THEOREM 6.4.11** (Interpretation of ZFC ). *If  $\varphi$  is a theorem of ZFC, then  $\varphi^W$  is provable in LNBG.*

*Proof.* For a formula  $\varphi(x_1, \dots, x_n)$  of ZFC,

$$\forall^W x_1, \dots, x_n (\varphi^W \rightarrow \Box \varphi^W)$$

is provable by Theorem 6.4.5(2), hence,

$$\forall^W x_1, \dots, x_n (\varphi^W \vee \neg \varphi^W)$$

is provable in LNBG. Now it suffices to show that for each nonlogical axiom  $A$  of ZFC,  $A^W$  is provable in LNBG.

(Equality axiom) $^W$  and (Extensionality) $^W$  are obvious.

(Pairing) $^W$ : By Theorem 6.4.5(6),

$$\text{ck}(u) \wedge \text{ck}(v) \rightarrow \text{ck}(\{u, v\}) \wedge \forall^W x (x \in \{u, v\} \leftrightarrow x = u \vee x = v).$$

(Union) $^W$ : Similarly.

(Power set) $^W$ : We have  $\forall^W u, x \left( x \in C(\mathcal{P}(u)) \leftrightarrow \forall^W t (t \in x \rightarrow t \in u) \right)$ .

( $\in$ -induction) $^W$ : By Theorem 6.4.10(3).

(Separation) $^W$ : If  $\text{ck}(u)$ , by Theorem 6.4.10(6),  $\text{ck}(\{x \in u \mid \varphi^W(x)\})$  and

$$\forall^W u, x \left( x \in \{x \in u \mid \varphi^W(x)\} \leftrightarrow x \in u \wedge \varphi^W(x) \right).$$

(Collection) $^W$ : Suppose  $\text{ck}(u) \wedge \forall^W x \in u \exists^W y \varphi^W(x, y)$ . By GA8(Colloction),

$$\exists v \forall x \in u \exists y \overset{\Box}{\in} v (\text{ck}(y) \wedge \varphi^W(x, y)).$$

Since  $y \overset{\Box}{\in} v \wedge \text{ck}(y) \rightarrow y \in C(v) \wedge \text{ck}(C(v))$ , we have

$$\exists^W v \forall^W x \in u \exists^W y \in v \varphi^W(x, y).$$

(Infinity) $^W$ : By Theorem 6.4.10(7).

(Choice) $^W$ , i.e.  $\forall^W u \exists^W f \forall^W x \in u [x \neq \emptyset \rightarrow \exists!^W y \in x (\langle x, y \rangle \in f)]$ , where  $x \neq \emptyset$  stands for  $\exists^W y (y \in x)$ . By Theorem 6.4.10(8). there exists an ordinal  $\alpha$  and a bijection  $\rho: \bigcup u \rightarrow \alpha$ . Define  $f: u \rightarrow \bigcup u$  by

$$f(x) = \rho^{-1} \left( \bigcap \{ \rho(t) \mid t \in x \} \right).$$

□

#### 6.4.4 Lattice valued model $W^{\mathcal{P}(1)}$ in $W$

The power set  $\mathcal{P}(1)$  of  $1 (= \{\emptyset\})$  is a global set, i.e.

$$\forall x (x \in \mathcal{P}(1) \rightarrow \Box(x \in \mathcal{P}(1))),$$

and a complete lattice with respect to the inclusion  $\subset$ . We write  $\leq$  instead of  $\subset$ . Then  $(\mathcal{P}(1), \leq)$  is a complete lattice. Let

$$(p \rightarrow q) = \{x \in 1 \mid 0 \in p \rightarrow 0 \in q\}, \quad \neg p = \{x \in 1 \mid \neg(0 \in p)\}.$$

$\rightarrow$  is the basic implication and  $\neg$  is the corresponding negation on  $\mathcal{P}(1)$ .

For a sentence  $\varphi$ , let

$$|\varphi| \stackrel{\text{def}}{=} \{t \in 1 \mid \varphi\}.$$

$|\varphi|$  is an element of  $\mathcal{P}(1)$ , and  $\varphi \iff 0 \in |\varphi|$ . Thus, the complete lattice  $\mathcal{P}(1)$  represents the truth value set of LNBG.

The relation  $\prec$  defined by

$$\alpha \prec \beta \stackrel{\text{def}}{\iff} \alpha, \beta \in \text{On} \wedge \alpha \in \beta$$

is a well founded relation and  $\text{Fld}(\prec) = \text{On}$ . Thus, the induction on  $\alpha \in \text{On}$  is justified in LNBG. Now we construct the  $\mathcal{P}(1)$ -valued model by induction on  $\alpha \in \text{On}$  as follows:

$$\begin{aligned} W_{\alpha}^{\mathcal{P}(1)} &= \{u \mid \exists \beta \in \alpha \exists \mathcal{D}u \subset W_{\beta}^{\mathcal{P}(1)} (\text{Gl}(\mathcal{D}u) \wedge u : \mathcal{D}u \rightarrow \mathcal{P}(1))\} \\ W^{\mathcal{P}(1)} &= \bigcup_{\alpha \in \text{On}} W_{\alpha}^{\mathcal{P}(1)} \end{aligned}$$

On  $W^{\mathcal{P}(1)}$ , the atomic relation  $=$  and  $\in$  are interpreted as

$$\begin{aligned} \llbracket x = y \rrbracket &= \bigwedge_{t \in \mathcal{D}x} (x(t) \rightarrow \llbracket t \in y \rrbracket) \wedge \bigwedge_{t \in \mathcal{D}y} (y(t) \rightarrow \llbracket t \in x \rrbracket) \\ \llbracket x \in y \rrbracket &= \bigvee_{t \in \mathcal{D}y} \llbracket x = t \rrbracket \wedge y(t). \end{aligned}$$

Logical operations  $\wedge, \vee, \rightarrow, \neg, \forall, \exists$  are interpreted as the correspondent operations on  $\mathcal{P}(1)$ . Then every sentence on  $W^{\mathcal{P}(1)}$  has its truth value in  $\mathcal{P}(1)$ , and we have

**THEOREM 6.4.12.** *For every sentence  $\varphi$ , “ $(0 \in \llbracket \varphi \rrbracket) \longleftrightarrow \varphi$ ” is provable in LNBG.*

*Proof.* We prove that there exists a global functional relation  $F$  such that:

- (i)  $\text{Dom}F = W^{\mathcal{P}(1)}$ , and
- (ii) for every formula  $\varphi(x_1, \dots, x_n)$  of LNBG on  $W^{\mathcal{P}(1)}$ ,

$$\llbracket \varphi(x_1, \dots, x_n) \rrbracket = |\varphi(F(x_1), \dots, F(x_n))|.$$

For  $x \in W^{\mathcal{P}(1)}$ , define  $F(x)$  by

$$F(x) = \{F(t) \mid t \in \mathcal{D}x \wedge 0 \in \llbracket t \in x \rrbracket\}.$$

Then we have:

- (1)  $0 \in \llbracket x = y \rrbracket \iff F(x) = F(y)$ ,  
 $0 \in \llbracket x \in y \rrbracket \iff F(x) \in F(y)$ .

- (2)  $\forall u \exists x (F(x) = u)$ .

Proof: Let  $\Psi(u) \stackrel{\text{def}}{\iff} \exists x (x \in W^{\mathcal{P}(1)} \wedge u = F(x))$ . Then by using GA8 (Collection) we have

$$\forall v (v \in u \rightarrow \Psi(v)) \implies \exists \alpha [\forall v (v \in u \rightarrow \exists y \in W_\alpha^{\mathcal{P}(1)} (v = F(y)))].$$

Let

$$\begin{cases} \mathcal{D}x = W_\alpha^{\mathcal{P}(1)} \\ x(y) = \{t \in 1 \mid F(y) \in u\} \end{cases}$$

Then  $x \in W^{\mathcal{P}(1)}$  and  $F(x) = u$ . Hence,  $\forall u \exists x (F(x) = u)$ .

- (3)  $0 \in \llbracket \varphi(x_1, \dots, x_n) \rrbracket \iff \varphi(F(x_1), \dots, F(x_n))$

Proof: We proceed by induction on the complexity of  $\varphi$ . If  $\varphi$  is atomic, then it is (1). If  $\varphi$  is of the form  $\varphi_1 \vee \varphi_2$ ,  $\varphi_1 \rightarrow \varphi_2$ ,  $\neg \varphi_1$  or  $\Box \varphi_1$ , then it follows from induction hypothesis. If  $\varphi(x_1, \dots, x_n)$  is of the form  $\forall x \psi(x, x_1, \dots, x_n)$ , then, by using (2),

$$\begin{aligned} 0 \in \llbracket \varphi \rrbracket &\iff 0 \in \bigwedge_x \llbracket \psi(x, x_1, \dots, x_n) \rrbracket \\ &\iff \forall x (\psi(F(x), F(x_1), \dots, F(x_n))) \\ &\iff \forall z \psi(z, F(x_1), \dots, F(x_n)). \end{aligned}$$

Similarly,

$$0 \in \llbracket \exists x \psi(x, x_1, \dots, x_n) \rrbracket \iff \exists z \psi(z, F(x_1), \dots, F(x_n)).$$

□

### 6.4.5 Completeness of LNBG

Now we will prove in LNBG that  $\mathcal{P}(1)$  is lattice-isomorphic to a complete lattice  $H$  which is a check set. (Theorem 6.4.8). As mentioned in the introduction, we mean by “a sentence  $\varphi$  of LNBG is valid” that

“ $\llbracket \varphi \rrbracket = 1$  on  $V^{\mathcal{L}}$  for all complete lattice  $\mathcal{L}$ ” is provable in ZFC.

Then the “completeness” of LNBG in the sense that every valid sentence of LNBG is provable in LNBG :

$$\text{ZFC} \vdash \text{“} \llbracket \varphi \rrbracket = 1 \text{ on } V^{\mathcal{L}} \text{ for all complete lattice } \mathcal{L} \text{”} \implies \text{LNBG} \vdash \varphi$$

can be proved (Theorem 6.4.14).

**THEOREM 6.4.13.** *There exists a complete lattice  $H$  which is a check set and a lattice-isomorphism  $\rho: \mathcal{P}(1) \rightarrow H$ .*

*Proof.* Since  $\mathcal{P}(1)$  is a global set, there exists a check set  $H$  together with a bijection  $\rho: \mathcal{P}(1) \rightarrow H$ , by Theorem (8). Define operations  $\bigwedge, \bigvee$  on  $H$  as follows:

$$\begin{aligned} \bigwedge A &= \rho\left(\bigcap_{a \in A} \rho^{-1}(a)\right), & \bigvee A &= \rho\left(\bigcup_{a \in A} \rho^{-1}(a)\right), \\ a \rightarrow b &= \begin{cases} 1, & \text{if } \rho^{-1}(a) \subset \rho^{-1}(b) \\ 0, & \text{if } \neg(\rho^{-1}(a) \subset \rho^{-1}(b)) \end{cases} \end{aligned}$$

for  $A \subset H$  such that  $\text{ck}(A)$ , and  $a, b \in H$ . Then  $\rho$  is a lattice -isomorphism.

□

**THEOREM 6.4.14** (“Completeness” of LNBG). *If a sentence  $\varphi$  is valid in every lattice-valued universe, then  $\varphi$  is provable in LNBG :*

$$\text{ZFC} \vdash \text{“} \llbracket \varphi \rrbracket = 1 \text{ on } V^{\mathcal{L}} \text{ for all complete lattice } \mathcal{L} \text{”} \implies \text{LNBG} \vdash \varphi$$

*Proof.* Suppose that a sentence  $\varphi$  is valid in every lattice valued universe. This means that  $(\llbracket \varphi \rrbracket = 1 \text{ on every lattice valued universe})$  is provable in our external universe of ZFC.

Since  $W$  is isomorphic to  $V$ ,

$$(\varphi \text{ is valid in every lattice valued universe})^W$$

is provable in LNBG. Let  $H \in W$  be a complete lattice with the basic implication which is lattice-isomorphic to  $\mathcal{P}(1)$ . That is, there exists a lattice-isomorphism  $\rho: \mathcal{P}(1) \rightarrow H$ . Construct the  $H$ -valued universe  $W^H$  in  $W$ . Then  $\llbracket \varphi \rrbracket = 1$  on  $W^H$ . It follows that  $\llbracket \varphi \rrbracket = 1$  on  $W^{\mathcal{P}(1)}$ , and  $\varphi$  is provable in LNBG by Theorem 6.4.12. ,  $\square$

By Theorem 6.4.14, a sentence  $\varphi$  holds in LNBG iff  $\llbracket \varphi \rrbracket = 1$ , on every lattice valued universe  $V^{\mathcal{L}}$ . Therefore, in order to discuss LNBG, it suffices to discuss the set theory on lattice valued universe.

## 6.5 Global Intuitionistic Set Theory

As seen in the Section 2.4, complete Heyting algebra is a counterpart of the intuitionistic logic, that is a complete lattice satisfying the distributive law:

$$a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i) \quad \text{on a complete lattice .}$$

Hence **global intuitionistic logic** GLJ is the lattice valued logic LL with the logical distributive law:

$$\textbf{Distributive law : } \varphi \wedge \exists x(\psi(x)) \leftrightarrow \exists x(\varphi \wedge \psi(x)),$$

$$\text{GLJ : LL + Distributive law}$$

Equivalent global intuitionistic logic  $\text{GLJ}_{\square}$  is obtained from LJ by introducing  $\square$  instead of the basic implication.

$$(a \rightarrow b) \stackrel{\text{def}}{=} \square(a \supset b) = \square\left(\bigvee\{c \in \mathcal{L} \mid c \wedge a \leq b\}\right).$$

A global intuitionistic set theory is the global intuitionistic logic with axioms of global set theory,

$$\text{GA1}, \dots, \text{GA11}$$

If we assume “ $\mathcal{P}(1)$  is a cHa”, i.e. “ $\mathcal{P}(1)$  is distributive”, in LNBG, then we have the distributive law of the logic:

$$\varphi \wedge \exists x \psi(x) \longleftrightarrow \exists x(\varphi \wedge \psi(x))$$

In fact, if  $\llbracket \varphi \rrbracket$  be the truth value of  $\varphi$  in  $W^{\mathcal{P}(1)}$ , then the following sentences are provable in LNBG.

$$\begin{aligned} \varphi \wedge \exists x \psi(x) &\leftrightarrow 0 \in \llbracket \varphi \wedge \exists x \psi(x) \rrbracket \\ &\leftrightarrow (0 \in \llbracket \varphi \rrbracket) \wedge (\bigvee_{x \in V^{\mathcal{P}(1)}} \llbracket \psi(x) \rrbracket) \\ &\leftrightarrow 0 \in \bigvee_{x \in V^{\mathcal{P}(1)}} \llbracket \varphi \wedge \psi(x) \rrbracket \\ &\leftrightarrow 0 \in \llbracket \exists x(\varphi \wedge \psi(x)) \rrbracket \\ &\leftrightarrow \exists x(\varphi \wedge \psi(x)) \end{aligned}$$

It follows that the intuitionistic implication  $\supset$  can be defined by

$$(\varphi \supset \psi) \stackrel{\text{def}}{\iff} 0 \in \bigcup \{u \in \mathcal{P}(1) \mid (\varphi \wedge (0 \in u)) \rightarrow \psi\}.$$

The corresponding logical implication is denoted by  $\rightarrow_{\text{I}}$ , and equality and membership relation are denoted by  $=_{\text{I}}$  and  $\in_{\text{I}}$ .

By the completeness of lattice valued set theory, the global intuitionistic set theory GINBG is also complete (Titani [14]).

$$\text{ZFC} \vdash “\llbracket \varphi \rrbracket = 1 \text{ on } V^{\Omega} \text{ for all Heyting algebra } \Omega” \implies \text{GINBG} \vdash \varphi.$$

## 6.6 Global classical set theory

Classical logic LK is a counterpart of complete Boolean algebra. That is, LK is an intuitionistic logic LJ with logical operation  $\neg$  satisfying

$$(\text{N1}) \vdash \neg \neg \varphi \Leftrightarrow \varphi,$$

$$(\text{N2}) \vdash \varphi \vee \neg \varphi, \quad \vdash \neg(\varphi \wedge \neg \varphi),$$

$$(N3) \quad \psi \rightarrow \varphi \implies \neg\varphi \rightarrow \neg\psi.$$

Complete Boolean algebra is a complete distributive lattice, i.e. Heyting algebra, with complementation  $\neg$  satisfying:

$$(N1) \quad \neg\neg a \leq a,$$

$$(N2) \quad a \vee \neg a = 1, \quad \neg(a \wedge b) = \neg a \vee \neg b,$$

$$(N3) \quad a \leq b \Leftrightarrow \neg b \leq \neg a.$$

The classical logic LK has an implication  $\supset$  defined by:

$$(a \supset b) \stackrel{\text{def}}{\iff} \neg a \vee b.$$

Thus, the basic implication can be defined by  $\supset$  and modal operator  $\Box$ . Another global classical logic  $GLK_{\Box}$  is obtained from LK by introducing  $\Box$  instead of the basic implication.

$$(a \rightarrow b) \stackrel{\text{def}}{=} \Box(a \supset b) = \Box(\neg a \vee b).$$

A global classical set theory is obtained from lattice valued set theory by adding logical operations and inference rules for them.

Nonlogical axioms of global classical set theory are axioms

$$GA1, \dots, GA11$$

of lattice set theory.

These axioms are valid on the lattice valued universe  $V^{\mathcal{L}}$ , hence on the Boolean valued universe  $V^B$ .

If we assume “ $\mathcal{P}(1)$  is a complete Boolean algebra cBa” in LNBG, then the logic satisfies the distributive law and (N1),(N2),(N3).

Hence we have:

**THEOREM 6.6.1.** *It is provable in LNBG + “ $\mathcal{P}(1)$  is a cBa” that the set theory is a classical set theory. It follows that GNBG is complete.*

*Proof.* (cf. [5]) For each axiom  $\varphi$  of classical set theory,

$$GNBG \vdash (\varphi \Leftrightarrow \llbracket \varphi \rrbracket = 1 \quad \text{on } V^{\mathcal{P}(1)}).$$

□

# Chapter 7

## Quantum set theory

Quantum set theory QNBG is developed by the quantum logic (cf.5.7 p.120) (QL or  $QL_{\square}$ ) with axioms  $GA_1, \dots, GA_{11}$  of the global set theory. QNBG is a set theory on a orthomodular-lattice valued universe.

Closed subspaces of a Hilbert space  $\mathcal{H}$  (or equivalently projections on  $\mathcal{H}$ ) form an orthomodular lattice, which is denoted by  $Q(\mathcal{H})$ . Unitary operator on  $\mathcal{H}$  is an automorphism and induces a symmetry on  $Q(\mathcal{H})$ .

Let  $\mathcal{U}$  be the set of unitary operators on  $\mathcal{H}$ . That is,  $Q(\mathcal{H})$  is an orthomodular lattice with symmetries  $\mathcal{U}$ .

The orthomodular lattice  $Q(\mathcal{H})$  has a structure of sheaf of Boolean lattice over  $\mathcal{U}$ . Thus,  $Q(\mathcal{H})$ -valued universe has the structure of sheaf of Boolean valued universe.

### 7.1 Complete orthomodular lattice $Q(\mathcal{H})$

Throughout this section, we assume that  $\mathcal{H}$  is a Hilbert space with a countable orthonormal basis :

$$\{\vec{e}_j\}_{j \in J} \text{ where } J = \{1, 2, \dots\}.$$

Elements of  $\mathcal{H}$  are expressed by vectors  $\vec{x}, \vec{y}, \dots$ , and the inner product of vectors  $\vec{x}, \vec{y}$  is denoted by  $(\vec{x}, \vec{y})$ . For vectors  $\vec{x}$  and  $\vec{y}$ ,  $\vec{x}$  is **orthogonal** to  $\vec{y}$ , in symbols  $\vec{x} \perp \vec{y}$ , if  $(\vec{x}, \vec{y}) = 0$ :

$$\vec{x} \perp \vec{y} \stackrel{\text{def}}{\iff} (\vec{x}, \vec{y}) = 0.$$

Let  $Q(\mathcal{H})$  be the complete orthomodular lattice consisting of closed subspaces of  $\mathcal{H}$ , where

$$\alpha \leq \beta \stackrel{\text{def}}{\iff} \alpha \subset \beta, \quad \alpha^\perp \stackrel{\text{def}}{=} \{\xi \in \mathcal{H} \mid \forall \psi \in \alpha (\xi \perp \psi)\} \quad \text{for } \alpha, \beta \in Q(\mathcal{H}).$$

If  $\{\alpha_i\}_{i \in I} \subset Q(\mathcal{H})$ , then  $\bigvee_{i \in I} \alpha_i$  is the supremum of  $\{\alpha_i\}_{i \in I}$  in  $Q(\mathcal{H})$  and  $\bigwedge_{i \in I} \alpha_i$  is the infimum of  $\{\alpha_i\}_{i \in I}$  in  $Q(\mathcal{H})$ . **Projection** is a bounded operator on  $\mathcal{H}$  which is self-adjoint and  $p^2 = p$ . Range of a projection  $p$ , denoted by  $\mathcal{R}(p)$ , is a closed subspace of  $\mathcal{H}$ :

$$\mathcal{R}(p) \stackrel{\text{def}}{=} \{p(\vec{x}) \mid \vec{x} \in \mathcal{H}\}$$

The set of range of projections forms a complete orthomodular lattice, isomorphic to  $Q(\mathcal{H})$  with respect to  $\leq$  and  $\perp$  defined by

$$p \leq q \stackrel{\text{def}}{\iff} \mathcal{R}(p) \leq \mathcal{R}(q), \quad \mathcal{R}(p^\perp) \stackrel{\text{def}}{=} \mathcal{R}(p)^\perp \quad \text{and} \quad p \circ q \stackrel{\text{def}}{\iff} \mathcal{R}(p) \circ \mathcal{R}(q).$$

Thus, we use the same notation  $Q(\mathcal{H})$  to denote the lattice of projections, as the orthomodular lattice of the ranges. The identity operator  $I$ ,  $\forall \vec{x} \in \mathcal{H} (I(\vec{x}) = \vec{x})$ , and zero operator  $0$ ,  $\forall \vec{x} \in \mathcal{H} (0(\vec{x}) = 0)$ , are members of  $Q(\mathcal{H})$ .

If  $\{p_\lambda\}_{\lambda \in \Lambda} \subset Q(\mathcal{H})$ , then

$$\bigvee_{\lambda \in \Lambda} p_\lambda = \sup\{p_\lambda \mid \lambda \in \Lambda\}, \quad \bigwedge_{\lambda \in \Lambda} p_\lambda = \inf\{p_\lambda \mid \lambda \in \Lambda\} \in Q(\mathcal{H}).$$

Let  $B$  be a maximal compatible subset of  $Q(\mathcal{H})$ . Then  $B$  is a complete Boolean lattice. If a self-adjoint operator  $\alpha$  has its spectral decomposition  $\alpha = \int \lambda dE_\lambda$ , where  $\{E_\lambda\}_\lambda \subset B$ , then  $\alpha$  is said to be a self-adjoint operator “in  $(B)$ ”.

Self-adjoint operators  $\alpha = \int \lambda dE_\lambda$  and  $\beta = \int \lambda dE_{\lambda'}$  are said to be **commutable** if for every pair  $\lambda, \lambda'$ ,

$$E_\lambda \cdot E_{\lambda'} = E_{\lambda'} \cdot E_\lambda,$$

If  $\alpha$  and  $\beta$  are bounded, then the commutativity of  $\alpha$  and  $\beta$  is equivalent to compatibility:

$$\alpha \cdot \beta = \beta \cdot \alpha \iff \alpha \circ \beta$$

**LEMMA 7.1.1.** *If  $\{\alpha_n\}_n$  is a set of self-adjoint, pairwise commutable operators, then there exists a complete Boolean lattice  $B$  of projections such that for every  $n$ ,  $\alpha_n$  is in  $(B)$ .*

Let  $\alpha$  and  $\beta$  be commutable self-adjoint operators. It is usual to define  $\alpha + \beta$  as the operator satisfying the conditions

$$\mathcal{D}(\alpha + \beta) = \mathcal{D}\alpha \cap \mathcal{D}\beta \quad \text{and}$$

$$\forall x \in \mathcal{D}(\alpha + \beta) ((\alpha + \beta)x = \alpha x + \beta x),$$

where  $\mathcal{D}(\alpha)$  denotes the domain of  $\alpha$ . The operator  $\alpha + \beta$ , defined in this way has a unique closed extension. For our purposes we define  $\alpha + \beta$  to be this unique closed extension. The operator  $\alpha + \beta$  is also self-adjoint. In the same way,  $\alpha \cdot \beta$  is defined to be the unique closed extension of the operator which maps  $x$ , with  $x \in \mathcal{D}(\beta)$  and  $\beta x \in \mathcal{D}(\alpha)$ , to  $\alpha\beta x$ . The operator  $\alpha \cdot \beta$  is also self-adjoint operator and  $\alpha \cdot \beta = \beta \cdot \alpha$ . Because of this definition, there is a possibility that  $\alpha + \beta$  and /or  $\alpha \cdot \beta$  is defined on the whole Hilbert space, and therefore bounded, even if  $\alpha$  and  $\beta$  are unbounded. In general, if the result of an operation is not closed but has a unique closed extension, we define  $O(\alpha, \beta)$  to be the unique closed extension of the result.

An operator  $\gamma$  is **normal**, if  $\gamma = \alpha + i\beta$  where  $\alpha$  and  $\beta$  are self-adjoint and commutable. Also  $\bar{\gamma} = \alpha - i\beta$  and  $\gamma\bar{\gamma} = \alpha^2 + \beta^2$ . Furthermore,  $\gamma$  is said to be in  $(B)$ , if  $\alpha$  and  $\beta$  are in  $(B)$ .

Let  $\alpha$  and  $\beta$  be self-adjoint and commutable. Then  $\alpha \leq \beta$  if and only if for every  $x \in \mathcal{D}(\alpha) \cap \mathcal{D}(\beta)$ ,  $(\alpha x, x) \leq (\beta x, x)$ .

## 7.2 $Q(\mathcal{H})$ -valued universe

$Q(\mathcal{H})$ -valued universe  $V^{Q(\mathcal{H})}$  is constructed by induction:

$$\begin{aligned} V_\alpha^{Q(\mathcal{H})} &= \{u \mid \exists \beta < \alpha \exists \mathcal{D}u \subset V_\beta^{Q(\mathcal{H})} (u : \mathcal{D}u \rightarrow Q(\mathcal{H}))\}, \\ V^{Q(\mathcal{H})} &= \bigcup_{\alpha \in On} V_\alpha^{Q(\mathcal{H})}. \end{aligned}$$

Truth values  $\llbracket u=v \rrbracket$  and  $\llbracket u \in v \rrbracket$  for  $u, v \in V^{Q(\mathcal{H})}$  are defined by induction on the rank of  $u, v$ .

$$\begin{aligned}\llbracket u=v \rrbracket &= \bigwedge_{x \in \mathcal{D}u} (u(x) \rightarrow \llbracket x \in v \rrbracket) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \rightarrow \llbracket x \in u \rrbracket) \\ \llbracket u \in v \rrbracket &= \bigvee_{x \in \mathcal{D}v} \llbracket u=x \rrbracket \wedge v(x).\end{aligned}$$

We call the set theory developed in the universe  $V^{Q(\mathcal{H})}$  **Hilbert quantum set theory**.

Observables in the quantum theory are represented by real numbers (i.e. Dedekind cuts) in the  $Q(\mathcal{H})$ -valued universe  $V^{Q(\mathcal{H})}$ .

In [11] and [12], G.Takeuti developed the quantum set theory on the  $Q(\mathcal{H})$ -valued universe  $V^{Q(\mathcal{H})}$ , where implication  $\rightarrow_{\mathbf{T}}$  is defined by

$$\varphi \rightarrow_{\mathbf{T}} \psi \stackrel{\text{def}}{\iff} \varphi^{\perp} \vee (\varphi \wedge \psi).$$

The corresponding equality  $=_{\mathbf{T}}$  and membership relation  $\in_{\mathbf{T}}$  are defined by

$$\begin{aligned}\llbracket u =_{\mathbf{T}} v \rrbracket &= \bigwedge_{x \in \mathcal{D}u} (u(x) \rightarrow_{\mathbf{T}} \llbracket x \in v \rrbracket) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \rightarrow_{\mathbf{T}} \llbracket x \in u \rrbracket) \\ \llbracket u \in_{\mathbf{T}} v \rrbracket &= \bigvee_{x \in \mathcal{D}v} \llbracket u =_{\mathbf{T}} x \rrbracket \wedge v(x),\end{aligned}$$

G.Takeuti showed in [11] that self-adjoint operators on  $\mathcal{H}$ , considered as observables, are represented as real numbers in  $V^{Q(\mathcal{H})}$ .

The operation  $\rightarrow_{\mathbf{T}}$  on  $Q(\mathcal{H})$  is an implication in the sense that

$$a \wedge (a \rightarrow_{\mathbf{T}} b) \leq b.$$

However,

$$'a \wedge b \leq c' \text{ does not imply } 'a \leq (b \rightarrow_{\mathbf{T}} c)' \text{ if } a, b \text{ are not compatible,}$$

because of non-distributivity of the lattice  $Q(\mathcal{H})$ . Consequently the transitivity of  $=_{\mathbf{T}}$  :

$$(u =_{\mathbf{T}} v) \wedge (v =_{\mathbf{T}} w) \rightarrow_{\mathbf{T}} (u =_{\mathbf{T}} w)$$

is not generally valid in the universe  $V^{Q(\mathcal{H})}$ . That is, the implication  $\rightarrow_{\mathbf{T}}$  is not strong enough to develop a set theory. Thus, we need a stronger implication  $\rightarrow$  for quantum logic.

A quantum set theory is developed in the universe  $V^{Q(\mathcal{H})}$ , using the local implication  $\rightarrow_{\mathbf{T}}$  together with basic implication, i.e. logical symbols :

$$\vee, \wedge, \perp, \forall, \exists, \square \text{ and } \rightarrow_{\mathbf{T}}.$$

Basic implication is defined by  $\square$  and  $\rightarrow_{\mathbf{T}}$ :

$$(a \rightarrow b) = \square(a \rightarrow_{\mathbf{T}} b) = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise.} \end{cases} \quad (7.2.1)$$

### 7.2.1 A sheaf structure of $Q(\mathcal{H})$

Let  $\mathfrak{e}_j$  be the subspace of  $\mathcal{H}$  spanned by singleton  $\{\vec{e}_j\}$ ,  $p_j$  be the corresponding projection onto  $\mathfrak{e}_j$ :

$$\mathfrak{e}_j \stackrel{\text{def}}{=} \{a_j \vec{e}_j \mid a_j \in \mathbb{C}\}; \quad \mathcal{H} = \bigvee_j \mathfrak{e}_j.$$

$$p_k(\sum_{j \in J} a_j \vec{e}_j) = a_k \vec{e}_k \text{ for } \{a_j\}_{j \in J} \subset \mathbb{C}; \quad p_k(\mathcal{H}) = \mathfrak{e}_k.$$

For each  $K \subset J$ , the supremum  $\bigvee_{j \in K} p_j$  of  $\{p_j\}_{j \in K}$  in  $Q(\mathcal{H})$  is the projection of  $\mathcal{H}$  onto the subspace spanned by  $\{\vec{e}_j\}_{j \in K}$ .

$$(\bigvee_{j \in K} p_j)(\sum_{j \in J} a_j \vec{e}_j) = \sum_{j \in K} a_j \vec{e}_j.$$

Let  $B \subset Q(\mathcal{H})$  be a maximal compatible subset of  $Q(\mathcal{H})$  defined by

$$B \stackrel{\text{def}}{=} \{\bigvee_{j \in K} p_j \mid K \subset J\},$$

$B$  is a complete Boolean algebra, which is isomorphic to the power set  $\mathcal{P}(J)$  of  $J$ .

$$\langle B, \wedge, \vee, \perp \rangle \cong \langle \mathcal{P}(J), \cap, \cup, {}^c \rangle$$

**DEFINITION 7.2.1.** A linear operator  $\sigma : \mathcal{H} \rightarrow \mathcal{H}$  is said to be **unitary** if

$$(\sigma \vec{x}, \sigma \vec{y}) = (\vec{x}, \vec{y}), \quad \text{for all } \vec{x}, \vec{y} \in \mathcal{H}.$$

Let  $\mathcal{U}$  be the set of all unitary operators on  $\mathcal{H}$ .

$$\mathcal{U} = \{\sigma : \mathcal{H} \rightarrow \mathcal{H} \mid \sigma \text{ is unitary} \}$$

$\mathcal{U}$  is a topological space. Let  $\mathcal{O}(\mathcal{U})$  be the set of open sets of  $\mathcal{U}$ .

Each unitary operator  $\sigma$  on  $\mathcal{H}$  induces an isomorphism  $\tilde{\sigma}$  on  $Q(\mathcal{H})$ :

$$\tilde{\sigma}(p) = \sigma p \sigma^{-1}.$$

$$\begin{aligned} \tilde{\sigma}(\bigvee_i p_i) &= \bigvee_i \tilde{\sigma}(p_i), & \tilde{\sigma}(\bigwedge_i p_i) &= \bigwedge_i \tilde{\sigma}(p_i), \\ \tilde{\sigma}(p^\perp) &= \tilde{\sigma}(p)^\perp, & \tilde{\sigma}(p \rightarrow q) &= (\tilde{\sigma}(p) \rightarrow \tilde{\sigma}(q)). \end{aligned}$$

In what follows we use the same notation  $\sigma$  instead for  $\tilde{\sigma}$ , i.e.

$$\sigma(p) = \{\sigma(a) \mid a \in p\} \text{ for closed subspace } p \subset \mathcal{H},$$

$$\sigma(p) = \sigma p \sigma^{-1} \text{ for projection } p : \mathcal{H} \rightarrow \mathcal{H}.$$

$\bigvee_{j \in K} \sigma(p_j)$  with  $K \subset J$  is an element of  $Q(\mathcal{H})$  spanned by  $\{\sigma(p_j)\}_{j \in K}$ .

Let  $B$  be a maximal compatible subset of  $Q(\mathcal{H})$ .  $B$  is a complete Boolean algebra isomorphic to  $\mathcal{P}(J)$ . If  $\sigma \in \mathcal{U}$ , then  $\sigma(B)$  is a complete Boolean algebra isomorphic to  $\mathcal{P}(J)$ .

For a unitary operator  $\sigma \in \mathcal{U}$  there exists a self-adjoint-operator  $A$  such that

$$\sigma = e^{iA}.$$

$e^{i(\pi/2)I}$  is a unitary operator such that  $e^{i(\pi/2)I}(\vec{e}_j) = i\vec{e}_j$  for  $j \in J$ , where  $I$  is the identity operator:  $Ix = x$  for  $x \in \mathcal{H}$ .  $e^{i(\pi/2)I}$  is also denoted by  $i$  in  $V^{Q(\mathcal{H})}$ .

$$e^{i(\pi/2)I}p = ip \text{ for } p \in Q(\mathcal{H}).$$

If  $p \in B$ , then

$$p \restriction q \iff (e^{i(\pi/2)I}p) \restriction q \text{ for } q \in Q(\mathcal{H}).$$

Hence,

$$p \in B \iff ip \in B.$$

Let  $f$  be the set of continuous functions  $f : U \rightarrow Q(\mathcal{H})$  such that

$$f(\sigma) \in \sigma(B) \text{ for } \sigma \in U.$$

For each  $U \in \mathcal{O}(\mathcal{U})$ ,

$f(U) = \{f : U \rightarrow Q(\mathcal{H}) \text{ continuous} \mid f(\sigma) \in \sigma(B), \sigma \in U\}$ , where

$$f \leq g \stackrel{\text{def}}{\iff} \forall \sigma \in U (f(\sigma) \leq g(\sigma)).$$

Then the set  $F(U)$  of all  $f(U)$  is a Boolean algebra, where

$$(f \wedge g)(\sigma) = f(\sigma) \wedge g(\sigma), \quad (f \vee g)(\sigma) = f(\sigma) \vee g(\sigma), \quad f^\perp(\sigma) = (f(\sigma))^\perp.$$

For  $U, V \in \mathcal{O}(X)$  such that  $V \subset U$  and  $f \in F(U)$ , let  $(f \upharpoonright V)$  be the restriction of  $f$  on  $V$ , i.e.  $(f \upharpoonright V)(\sigma) = f(\sigma)$  for  $\sigma \in V$ , and let

$$r_{V,U}(f) \stackrel{\text{def}}{=} f \upharpoonright V$$

Then

$$r_{V,U}(F(U)) = F(V) \quad \text{for } U, V \in \mathcal{O}(X) \text{ such that } V \subset U \quad \text{and}$$

$\langle F, r \rangle$  is a sheaf of Boolean algebra over  $\mathcal{U}$ ,  $\text{Sh}_{\mathcal{U}} B$ .

### 7.3 Sheaf structure of $V^{Q(\mathcal{H})}$

In the previous section, orthomodular lattice  $Q(\mathcal{H})$  was represented as a sheaf  $\langle F, r \rangle$  of complete Boolean algebra over a topological space  $\langle \mathcal{U}, \mathcal{O}(\mathcal{U}) \rangle$ .

Now we extend the sheaf  $\langle F, r \rangle$  to a sheaf of Boolean valued universe  $\langle \mathcal{F}, r \rangle$ , where :

- (1) For  $U \in \mathcal{O}(\mathcal{H})$ ,  $F(U)$  is a complete Boolean algebra and  $V^{F(U)}$  is a Boolean valued universe, where

$$v \in V_\alpha^{F(U)} \iff \begin{cases} \mathcal{D}v \subset V_{<\alpha}^{F(U)} \\ v(x) \in F(U). \end{cases} \quad V^{F(U)} \stackrel{\text{def}}{=} \bigcup_{\alpha \in \text{Ord}} V_\alpha^{F(U)}$$

- (2)  $\mathcal{F}$  is a mapping which associates a Boolean valued universe  $V^{F(U)}$  to each  $U \in \mathcal{O}(X)$ .

$$\mathcal{F} : U \mapsto V^{F(U)} \quad \text{for } U \in \mathcal{O}(X)$$

(3) If  $U, W \in \mathcal{O}(X)$  and  $U \subset W$ , then  $r_{U,W} : V^{F(W)} \rightarrow V^{F(U)}$ , where

$$\mathcal{D}(r_{U,W}(u)) = \{r_{U,W}(t) \mid t \in \mathcal{D}u\} \quad r_{U,W}(u)(r_{U,W}(t)) = r_{U,W}(u(t))$$

(4)  $r_{U,W} : V^{F(W)} \rightarrow V^{F(U)}$  is a homomorphism.

(5)  $\sigma \in \mathcal{U}$  is extended to  $\sigma : V^{Q(\mathcal{H})} \rightarrow V^{Q(\mathcal{H})}$  by:

$$\mathcal{D}\sigma(u) = \{\sigma(x) \mid x \in \mathcal{D}u\}, \quad \sigma(u)(\sigma(x)) = \sigma(u(x)) \quad \text{for } u \in V^{Q(\mathcal{H})}.$$

(6)  $\sigma : V^{Q(\mathcal{H})} \rightarrow V^{Q(\mathcal{H})}$  is an automorphism, i.e. for a formula  $\varphi(u_1, \dots, u_n)$  of QNBG

$$\sigma \llbracket \varphi(u_1, \dots, u_n) \rrbracket = \llbracket \varphi(\sigma(u_1), \dots, \sigma(u_n)) \rrbracket.$$

*Proof.* (a)  $\sigma(\llbracket u=v \rrbracket) = \llbracket \sigma(u)=\sigma(v) \rrbracket$ ,  $\sigma(\llbracket u \in v \rrbracket) = \llbracket \sigma(u) \in \sigma(v) \rrbracket$ .

$\therefore$ ) Assume that, for  $x, y \in V_{<\alpha}^{Q(\mathcal{H})}$ ,

$$\llbracket \sigma(x)=\sigma(y) \rrbracket = \sigma(\llbracket x=y \rrbracket).$$

Then for and  $u, v \in V_{\alpha}^{Q(\mathcal{H})}$ ,

$$\llbracket \sigma(x) \in \sigma(v) \rrbracket = \bigvee_{y \in \mathcal{D}v} (\llbracket \sigma(x) = \sigma(y) \rrbracket \wedge$$

$$\sigma(v)(\sigma(y)))$$

$$= \sigma \left( \bigvee_{y \in \mathcal{D}v} \llbracket x=y \rrbracket \wedge v(y) \right) = \sigma \llbracket x \in v \rrbracket.$$

Similarly,

$$\llbracket \sigma(y) \in \sigma(u) \rrbracket = \bigvee_{x \in \mathcal{D}u} \llbracket \sigma(y) = \sigma(x) \rrbracket \wedge \sigma(u)(\sigma(x)) = \sigma \llbracket y \in u \rrbracket$$

$$\therefore \sigma \llbracket u=v \rrbracket = \bigwedge_{x \in \mathcal{D}u} (\sigma(u)(\sigma(x)) \rightarrow \sigma \llbracket x \in v \rrbracket)$$

$$\wedge \bigwedge_{y \in \mathcal{D}v} (\sigma(v)(\sigma(y)) \rightarrow \sigma \llbracket y \in u \rrbracket)$$

$$= \llbracket \sigma(u) = \sigma(v) \rrbracket.$$

$$\sigma \llbracket u \in v \rrbracket = \left( \bigvee_{y \in \mathcal{D}v} \sigma \llbracket u=y \rrbracket \wedge \sigma(v(y)) \right) = \bigvee_{y \in \mathcal{D}v} \llbracket \sigma(u) \in \sigma(v) \rrbracket.$$

(b)  $\sigma$  preserves  $\wedge, \vee, ^\perp, \rightarrow, \forall$  and  $\exists$ . Therefore,

$$\sigma \llbracket \varphi(u_1, \dots, u_n) \rrbracket = \llbracket \varphi(\sigma(u_1), \dots, \sigma(u_n)) \rrbracket.$$

□

For  $U \in \mathcal{O}(\mathcal{U})$ , let

$$f(U) = \{f : U \rightarrow V^{Q(\mathcal{H})} \text{ continuous} \mid \forall \sigma \in U (f(\sigma) \in V^{\sigma(B)})\} \quad \text{and}$$

let  $\mathcal{F}(U)$  be the set of  $f(U)$ .

For  $f(U), g(U) \in \mathcal{F}(U)$ , let

$$f(U) \in g(U) \stackrel{\text{def}}{\iff} \forall \sigma \in U (f(\sigma) \in g(\sigma)), \quad f(U) = g(U) \stackrel{\text{def}}{\iff} \forall \sigma \in U (f(\sigma) = g(\sigma)).$$

Then the set  $\mathcal{F}(U)$  of all  $f(U)$  is a Boolean valued universe. For  $U, V \in \mathcal{O}(X)$  such that  $V \subset U$  and  $f \in \mathcal{F}(U)$ , let  $(\mathcal{F}(U) \upharpoonright V)$  be the set of restriction on  $V$ , and let

$$r_{V,U}(f(U)) \stackrel{\text{def}}{=} f(U) \upharpoonright V$$

Then

$$r_{V,U}(\mathcal{F}(U)) = \mathcal{F}(V) \quad \text{for } U, V \in \mathcal{O}(X) \text{ such that } V \subset U \quad \text{and}$$

$\langle \mathcal{F}, r \rangle$  is a sheaf of Boolean valued universe over  $\mathcal{U}, \text{Sh}_{\mathcal{U}} V^B$ .

## 7.4 Quantum numbers

The set  $\omega$  of all natural numbers is constructed from the empty set by the successor function  $x \mapsto x \cup \{x\}$ .

$$n \in \omega \iff n = 0 \vee \exists m \in n (n = m \cup \{m\}).$$

The empty set is a check set which we denote by  $\check{0}$ .

$$\check{0} = \emptyset.$$

If  $m$  is a check set then  $m \cup \{m\}$  is also a check set by Theorem 6.4.10 (6). Thus,  $(m + 1)^\sim = \check{m} \cup \{\check{m}\}$ . Therefore, the set of natural numbers in  $V^{Q(\mathcal{H})}$  is  $\check{\omega}$ , where  $\mathcal{D}\check{\omega} = \{\check{n} \mid n \in \omega\}$  and  $\check{\omega}(\check{n}) = 1$ . For convenience we write

$$\check{\omega} = \{\check{n} \mid n \in \omega\}.$$

For  $m, n, m', n' \in \omega$ , let

$$\langle \check{m}, \check{n} \rangle \equiv \langle \check{m}', \check{n}' \rangle \stackrel{\text{def}}{\iff} \check{m} + \check{n}' = \check{m}' + \check{n}.$$

$\equiv$  is an equivalence relation on the set  $\check{\omega} \times \check{\omega}$  of pairs of natural numbers. The integers are equivalence classes of pairs of natural numbers.

Since the relation  $\equiv$  is defined by  $\square$ -closed formula, the equivalence class of  $\mathbb{N}$  by  $\equiv$  is a check set according to Theorem 6.4.10. The equivalence class of  $\langle \check{m}, \check{n} \rangle$  is denoted by  $\check{m} - \check{n}$ . The set of integers in  $V^{Q(\mathcal{H})}$  is the check set associated with  $\mathbb{Z}$ .

$$\check{\mathbb{Z}} = (\check{\omega} / \equiv) = \{\check{m} - \check{n} \mid m, n \in \omega\} = \{(m - n)^\sim \mid m, n \in \omega\}.$$

The rational numbers are constructed as equivalence classes of pairs of integers :

$$\langle \check{a}, \check{b} \rangle \equiv \langle \check{a}', \check{b}' \rangle \stackrel{\text{def}}{\iff} \check{a} \cdot \check{b}' = \check{a}' \cdot \check{b} \quad \text{for } a, b, a', b' \in \mathbb{Z}.$$

The equivalence class, denoted by  $\check{a}/\check{b}$ , is a rational number. The set of rational numbers in  $V^{Q(\mathcal{H})}$  is the check set of  $\mathbb{Q}$ .

$$\check{\mathbb{Q}} = (\check{\mathbb{Z}} / \equiv) = \{\check{a}/\check{b} \mid a, b \in \mathbb{Z}\}.$$

Real numbers, defined as Dedekind's cuts of rational numbers, are not check sets in contrast with integers and rational numbers. Dedekind cut of  $\check{\mathbb{Q}}$  is a subset of  $\check{\mathbb{Q}}$ , which is not necessarily a check set. The set  $\mathcal{P}(\check{\mathbb{Q}})$  of all subsets of  $\check{\mathbb{Q}}$  is defined by

$$\mathcal{DP}(\check{\mathbb{Q}}) = \{\alpha \mid \alpha \subset \check{\mathbb{Q}}\}, \quad \mathcal{P}(\check{\mathbb{Q}})(\alpha) = 1.$$

**DEFINITION 7.4.1.**  $\alpha \in \mathcal{P}(\check{\mathbb{Q}})$  is called a **quantum real** if

$$\begin{aligned} (D1) \quad & \exists x \in \check{\mathbb{Q}}(x \in \alpha) \wedge \exists x \in \check{\mathbb{Q}}(x \in \alpha)^\perp, \\ (D2) \quad & \forall x \in \check{\mathbb{Q}} \left( (x \in \alpha) \longleftrightarrow \forall y \in \check{\mathbb{Q}}(x < y \rightarrow_{\mathbf{T}} y \in \alpha) \right). \end{aligned}$$

The set of quantum reals in  $V^{Q(\mathcal{H})}$  will be denoted by  $\mathfrak{R}^{Q(\mathcal{H})}$ .

**LEMMA 7.4.1.** If  $\llbracket \alpha \subset \check{\mathbb{Q}} \rrbracket = 1$  in  $V^{Q(\mathcal{H})}$ , where  $\alpha \subset \beta \stackrel{\text{def}}{\longleftrightarrow} \forall x(x \in \alpha \rightarrow x \in \beta)$ , then there exists  $v$  in  $V^{Q(\mathcal{H})}$  such that  $\mathcal{D}v = \{\check{r} \mid r \in \mathbb{Q}\} (= \mathcal{D}\check{\mathbb{Q}})$  and  $\llbracket \alpha = v \rrbracket = 1$ .

*Proof.* Note that  $\llbracket x = y \rrbracket \in \mathbf{2}$ .

$$\llbracket x = y \rrbracket \wedge \bigvee_j a_j = \bigvee_j \llbracket x = y \rrbracket \wedge a_j \quad \text{for } \forall \{a_i\} \subset Q.$$

Define  $v$  by  $\mathcal{D}v = \{\check{r} \mid r \in \mathbb{Q}\}$ , and  $v(\check{r}) = \llbracket \check{r} \in \alpha \rrbracket$  for  $r \in \mathbb{Q}$ . If  $x \in \mathcal{D}\alpha$ ,

$$\begin{aligned} \alpha(x) &\leq \llbracket x \in \check{\mathbb{Q}} \rrbracket \wedge \llbracket x \in \alpha \rrbracket = \bigvee_{r \in \mathbb{Q}} (\llbracket x = \check{r} \rrbracket \wedge \llbracket x \in \alpha \rrbracket) \\ &= \bigvee_{r \in \mathbb{Q}} (\llbracket x = \check{r} \rrbracket \wedge \llbracket \check{r} \in \alpha \rrbracket) \leq \llbracket x \in v \rrbracket \end{aligned}$$

Therefore,  $\llbracket \alpha = v \rrbracket = 1$ . □

We continue to fix a basis  $\{\vec{e}_j\}_{j \in J}$  of  $\mathcal{H}$ , and let  $p_j$  be the projection onto the subspace spanned by singleton  $\{\vec{e}_j\}$ .

$$p_j(x) = (x, \vec{e}_j) \cdot \vec{e}_j \quad \text{for } x \in \mathcal{H}.$$

**DEFINITION 7.4.2.** An element  $p$  of  $Q(\mathcal{H})$  is called an **atom** if

$$(p \neq 0) \quad \text{and} \quad ((0 \leq q \leq p) \leftrightarrow (0 = q \quad \text{or} \quad q = p)).$$

Each  $p_j$  is an atom. The maximal compatible set  $\{p_j\}_{j \in J}$  of atoms is a **basis** of  $Q(\mathcal{H})$  such that

$$p_j \perp p_k \text{ for all } j, k \in J \quad \text{and} \quad \bigvee_j p_j = 1.$$

Another basis of  $\mathcal{H}$  is  $\{\sigma(\vec{e}_j)\}_{j \in J}$  for some unitary operation  $\sigma$ , and the corresponding basis of  $Q(\mathcal{H})$  is

$$\{\sigma p_j \sigma^{-1}\}_{j \in J}.$$

**THEOREM 7.4.2.** *If  $u$  is a quantum real in  $V^{Q(\mathcal{H})}$ , that is,*

$$\llbracket u \text{ satisfies } D1 \text{ and } D2 \rrbracket = 1,$$

*then  $\{\llbracket \check{r} \in u \rrbracket \mid r \in \mathbb{Q}\}$  is compatible.*

*Proof.* Because of

$$r \leq s \implies \llbracket \check{r} \in u \rrbracket \leq \llbracket \check{s} \in u \rrbracket.$$

□

Therefore, there exists a unitary operator  $\sigma \in \mathcal{U}$  such that

$$\{\sigma(p_j)\}_{j \in J} \text{ where } J = \{1, 2, \dots\}$$

is an orthonormal basis of  $Q(\mathcal{H})$ , and each  $\llbracket \check{r} \in u \rrbracket$  ( $r \in \mathbb{Q}$ ) is spanned by a subset of the basis, where  $\sigma(p_j) = \sigma p_j \sigma^{-1}$ . That is,

**THEOREM 7.4.3.** *If  $u$  is a quantum real, then*

$$\{\llbracket \check{r} \in u \rrbracket \mid r \in \mathbb{Q}\}$$

*is a compatible subset of  $Q(\mathcal{H})$  such that*

$$\llbracket \check{r} \in u \rrbracket \leq \llbracket \check{s} \in u \rrbracket \text{ if } r \leq s \text{ and } \bigvee_{r \in \mathbb{Q}} \llbracket \check{r} \in u \rrbracket = 1.$$

*Hence, there exists a basis  $\{\sigma(e_j)\}_{j \in J}$  of  $\mathcal{H}$  such that every  $\llbracket \check{r} \in u \rrbracket$  is spanned by a subset of  $\{\sigma(e_j)\}_{j \in J}$ .*

**THEOREM 7.4.4** (cf. [11]). *If  $u$  is a quantum real in  $V^{Q(\mathcal{H})}$ , then  $E_u: \mathbb{R} \rightarrow Q(\mathcal{H})$  defined by*

$$E_u(\lambda) = \bigwedge_{\lambda < r} \llbracket \check{r} \in u \rrbracket$$

*is a resolution of the identity, i.e.  $E_u$  satisfies*

$$\bigwedge_{\lambda \in \mathbb{R}} E_u(\lambda) = 0 \quad \bigvee_{\lambda \in \mathbb{R}} E_u(\lambda) = 1 \quad E_u(\lambda) = \bigwedge_{\lambda < \mu} E_u(\mu).$$

Hence, we have the spectral representation of the corresponding self-adjoint operator on the Hilbert space  $\mathcal{H}$ :

**THEOREM 7.4.5.** *Quantum real  $\alpha$  in  $V^{Q(\mathcal{H})}$  represents a self-adjoint operator on  $\mathcal{H}$ .*

$$A = \int \lambda dE_\alpha(\lambda),$$

*Conversely, if  $A$  is a positive self-adjoint operator on  $\mathcal{H}$ ,*

$$A = \int \lambda dE(\lambda)$$

*then  $\alpha \in V^{Q(\mathcal{H})}$  defined by  $\mathcal{D}\alpha = \{\check{r} \mid r \in \mathbb{Q}\}$  and  $\alpha(\check{r}) = \bigwedge_{r < s} E(s)$  is a Dedekind cut in  $V^{Q(\mathcal{H})}$ , and*

$$A = \int \lambda dE_\alpha(\lambda).$$

### 7.4.1 Representation of projections in $V^{Q(\mathcal{H})}$

A projection on  $\mathcal{H}$  is represented as quantum real in  $V^{Q(\mathcal{H})}$ . Let  $u$  be a quantum real represented by the proposition  $p$  and  $A_u$  be the corresponding self-adjoint operator :  $A_u = \int \lambda dE_u(\lambda)$ . Then

$$\int \lambda dE_u(\lambda) = 0 \cdot E_u(0) + 1 \cdot (E_u(1) - E_u(0)) = E_u(0)^\perp.$$

Then  $E_u(0)^\perp = p$ . For, if self-adjoint operator  $\int \lambda dE_u(\lambda)$  is a projection, then

$$u(\check{r}) = \bigwedge_{r < s} E_u(s) = \begin{cases} 0, & \text{if } r < 0, \\ p^\perp, & \text{if } 0 \leq r < 1, \\ 1, & \text{if } 1 \leq r, \end{cases}$$

which is equivalent to

$$\llbracket u =_{\mathbf{T}} \check{1} \rrbracket = \llbracket u =_{\mathbf{T}} \check{0} \rrbracket^\perp = p.$$

Thus, projection is characterized as a real number  $u$  such that  $\llbracket u =_{\mathbf{T}} \check{1} \rrbracket = \llbracket u =_{\mathbf{T}} \check{0} \rrbracket^\perp$ .

**DEFINITION 7.4.3.** *For each  $p \in Q(\mathcal{H})$ , we define  $\hat{p}$  in  $V^{Q(\mathcal{H})}$  by*

$$\mathcal{D}\hat{p} = \mathcal{D}\check{Q}, \quad \hat{p}(\check{r}) = \begin{cases} 0, & \text{if } r < 0, \\ p^\perp, & \text{if } 0 \leq r < 1, \\ 1, & \text{if } 1 \leq r \end{cases}$$

**THEOREM 7.4.6.** *If  $p \in Q(\mathcal{H})$ , then  $\widehat{p} \in V^{Q(\mathcal{H})}$  satisfies*

$$\llbracket \widehat{p} =_{\mathbf{T}} \check{1} \rrbracket = \llbracket \widehat{p} =_{\mathbf{T}} \check{0} \rrbracket^{\perp} = p.$$

*Conversely, if a Dedekind cut  $u$  in  $V^{Q(\mathcal{H})}$  satisfies*

$$\llbracket u =_{\mathbf{T}} \check{1} \rrbracket = \llbracket u =_{\mathbf{T}} \check{0} \rrbracket^{\perp} = p,$$

*then  $\llbracket u = \widehat{p} \subset 1 \rrbracket = 1$ . It follows that  $p \in Q(\mathcal{H})$  is the range of projection  $A_{\widehat{p}}$ .*

*Proof.* If  $p \in Q(\mathcal{H})$ , then

$$\begin{aligned} (\check{r} \in \widehat{p} \rightarrow_{\mathbf{T}} \check{r} \in \check{1}) \wedge (\check{r} \in \check{1} \rightarrow_{\mathbf{T}} \check{r} \in \widehat{p}) &= \begin{cases} 1, & r < 0, \\ p, & 0 \leq r < 1, \\ 1, & 1 \leq r, \end{cases} \\ (\check{r} \in \widehat{p} \rightarrow_{\mathbf{T}} \check{r} \in \check{0}) \wedge (\check{r} \in \check{0} \rightarrow_{\mathbf{T}} \check{r} \in \widehat{p}) &= \begin{cases} 1, & r < 0, \\ p^{\perp}, & 0 \leq r < 1, \\ 1, & 1 \leq r. \end{cases} \end{aligned}$$

Therefore,  $\llbracket \widehat{p} =_{\mathbf{T}} \check{1} \rrbracket = \llbracket \widehat{p} =_{\mathbf{T}} \check{0} \rrbracket^{\perp} = p$ .

Conversely, let  $u$  be a positive real in  $V^{Q(\mathcal{H})}$  such that

$$\llbracket u =_{\mathbf{T}} \check{1} \rrbracket = \llbracket u =_{\mathbf{T}} \check{0} \rrbracket^{\perp} = p.$$

We show  $\llbracket u = \widehat{p} \rrbracket = 1$ . Since  $\mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\}$ , it suffices to show that

$$u(\check{r}) = \begin{cases} 0, & \text{if } r < 0, \\ p^{\perp}, & \text{if } 0 \leq r < 1, \\ 1, & \text{if } 1 \leq r. \end{cases}$$

If  $1 \leq r$ , then

$$p \leq (\check{1}(\check{r}) \rightarrow_{\mathbf{T}} \llbracket \check{r} \in u \rrbracket) \leq u(\check{r})$$

and

$$p^{\perp} \leq (\check{0}(\check{r}) \rightarrow_{\mathbf{T}} \llbracket \check{r} \in u \rrbracket) \leq u(\check{r})$$

It follows that  $u(\check{r}) \geq p \vee p^{\perp} = 1$ . Similarly, if  $0 \leq r < 1$ , then  $p \leq u(\check{r})^{\perp}$  and  $p^{\perp} \leq u(\check{r})$ . Hence,  $u(\check{r}) = p^{\perp}$ . If  $r < 0$ , then  $p \leq u(\check{r})^{\perp}$  and  $p^{\perp} \leq u(\check{r})^{\perp}$ . Hence,  $u(\check{r}) \leq p \wedge p^{\perp} = 0$  □

**THEOREM 7.4.7.** *If  $\sigma \in \mathcal{U}$  and  $p \in Q(\mathcal{H})$ , then  $\sigma : Q(\mathcal{H}) \rightarrow Q(\mathcal{H})$  is a bijection preserving  $\vee$  and  $^\perp$ , and*

$$\llbracket \widehat{\sigma(p)} = \sigma(\widehat{p}) \rrbracket = 1 \quad \text{for } p \in Q(\mathcal{H}).$$

*Proof.*

$$\llbracket \widehat{\sigma(p)} =_{\mathbf{T}} \sigma(\check{1}) \rrbracket = \llbracket \widehat{\sigma(p)} =_{\mathbf{T}} \sigma(\check{0}) \rrbracket^\perp = \sigma(p).$$

Hence, by Theorem 7.4.6,  $\llbracket \sigma(\widehat{p}) = \widehat{\sigma(p)} \rrbracket = 1$ .

□

**DEFINITION 7.4.4** (in  $V^{Q(\mathcal{H})}$ ). *A real number  $u$  is called a **projection** if*

$$u =_{\mathbf{T}} \check{1} \leftrightarrow (u =_{\mathbf{T}} \check{0})^\perp.$$

*Projections  $u, v$  in  $V^{q(\mathcal{H})}$  are said to be **orthogonal** if*

$$u =_{\mathbf{T}} \check{1} \leftrightarrow (v =_{\mathbf{T}} \check{1})^\perp.$$

**LEMMA 7.4.8.** *For  $p, q \in Q(\mathcal{H})$ ,  $p \leq q \iff \llbracket \widehat{p} \leq \widehat{q} \rrbracket = 1$ .*

*Proof.* For  $r \in \mathbb{Q}$ ,

$$r < 0 \vee 1 \leq r \Rightarrow \llbracket \check{r} \in \widehat{p} \rrbracket = \llbracket \check{r} \in \widehat{q} \rrbracket,$$

$$0 < r \leq 1 \Rightarrow p \leq q \Leftrightarrow \llbracket \check{r} \in \widehat{q} \rrbracket = q^\perp \leq p^\perp = \llbracket \check{r} \in \widehat{p} \rrbracket.$$

□

## 7.4.2 Operations on real numbers in $V^{Q(\mathcal{H})}$

**DEFINITION 7.4.5.** *For quantum reals  $u, v$  in  $V^{Q(\mathcal{H})}$ ,*

$$u \leq v \stackrel{\text{def}}{\iff} \forall r \in \mathbb{Q} (r \in v \rightarrow r \in u)$$

*$u$  is said to be **positive**, if  $\check{0} \leq u$ .*

$$u \leq_{\mathbf{T}} v \stackrel{\text{def}}{\iff} \forall r \in \mathbb{Q} (r \in v \rightarrow_{\mathbf{T}} r \in u)$$

**DEFINITION 7.4.6.** For quantum reals  $u, v$  in  $V^{Q(\mathcal{H})}$ ,  $u + v$  is defined by

$$u + v \stackrel{\text{def}}{=} \{r \in \check{\mathbb{Q}} \mid \forall s \in \check{\mathbb{Q}} (r < s \rightarrow_{\mathbf{T}} \exists r_1, r_2 \in \check{\mathbb{Q}} (s = r_1 + r_2 \wedge r_1 \in u \wedge r_2 \in v))\}.$$

For positive quantum reals  $u, v$  in  $V^{Q(\mathcal{H})}$ ,  $u \cdot v$  is defined by

$$u \cdot v \stackrel{\text{def}}{=} \{r \in \check{\mathbb{Q}} \mid \forall s \in \check{\mathbb{Q}} (r < s \rightarrow_{\mathbf{T}} \exists r_1, r_2 \in \check{\mathbb{Q}} (s = r_1 \cdot r_2 \wedge r_1 \in u \wedge r_2 \in v))\}.$$

**LEMMA 7.4.9.** Suppose that  $B$  is a complete Boolean sublattice of  $Q(\mathcal{H})$ ,  $u, v$  are quantum reals in  $V^B$ , and  $p \in B$ . Then

$$\llbracket u[p + v[p = (u + v)[p]] = 1 \rrbracket = \llbracket u[p \cdot v[p = (u \cdot v)[p]] = 1 \rrbracket = 1.$$

*Proof.*

$$\begin{aligned} ((u + v)[p](\check{r})) &= \left( \bigwedge_{\substack{s \in \check{\mathbb{Q}} \\ r < s}} \llbracket \exists r_1, r_2 \in \check{\mathbb{Q}} (s = r_1 + r_2 \wedge r_1 \in u \wedge r_2 \in u) \rrbracket \right) \wedge p \\ &= \left[ \bigwedge_{\substack{s \in \check{\mathbb{Q}} \\ r < s}} \left( \bigvee_{\substack{r_1, r_2 \in \check{\mathbb{Q}} \\ s = r_1 + r_2}} u(r_1) \wedge v(r_2) \right) \right] \wedge p \\ &= \bigwedge_{\substack{s \in \check{\mathbb{Q}} \\ r < s}} \left( \bigvee_{\substack{r_1, r_2 \in \check{\mathbb{Q}} \\ s = r_1 + r_2}} u[p(r_1)] \wedge v[p(r_2)] \right) \\ &= (u[p + v[p]])(\check{r}). \end{aligned}$$

$\llbracket u[p \cdot v[p = (u \cdot v)[p]] = 1 \rrbracket = 1$  is proved similarly.  $\square$

**DEFINITION 7.4.7.** Let  $u, v$  be real numbers in  $V^{Q(\mathcal{H})}$ .  $u, v$  are said to be **compatible**, if  $\{\llbracket \check{r} \in u \rrbracket \mid r \in \mathbb{Q}\} \cup \{\llbracket \check{r} \in v \rrbracket \mid r \in \mathbb{Q}\}$  is compatible.

**LEMMA 7.4.10.**  $u, v$  are compatible real numbers in  $V^{Q(\mathcal{H})}$  if and only if there exists a complete Boolean sublattice  $B$  of  $Q(\mathcal{H})$  such that  $u, v \in V^B$ .

*Proof.*  $M = \{\llbracket \check{r} \in u \rrbracket \mid r \in \mathbb{Q}\} \cup \{\llbracket \check{r} \in v \rrbracket \mid r \in \mathbb{Q}\}$  is compatible and  $M \subset Q(\mathcal{H})$ . Hence,  $B = M''$  is a Boolean sublattice of  $Q(\mathcal{H})$  such that  $u, v \in V^B$ , where  $A' = \{p \in Q(\mathcal{H}) \mid p \restriction a \text{ for all } a \in A\}$ .  $\square$

**THEOREM 7.4.11** ([11]). Let  $B \subset Q(\mathcal{H})$  be a complete Boolean sublattice of  $Q(\mathcal{H})$ . If  $u, v$  are quantum reals in  $V^B$ , then  $u + v$  and  $u \cdot v$  are quantum reals in  $V^B$ .

If  $u_1, u_2, v_1, v_2$  are quantum reals in  $V^B$ , then

$$\llbracket u_1 =_{\mathbf{T}} u_2 \rrbracket \wedge \llbracket v_1 =_{\mathbf{T}} v_2 \rrbracket \leq \llbracket u_1 + v_1 =_{\mathbf{T}} u_2 + v_2 \rrbracket,$$

$$\llbracket u_1 =_{\mathbf{T}} u_2 \rrbracket \wedge \llbracket v_1 =_{\mathbf{T}} v_2 \rrbracket \leq \llbracket u_1 \cdot v_1 =_{\mathbf{T}} u_2 \cdot v_2 \rrbracket.$$

*Proof.* Let  $\llbracket u_1 =_{\mathbf{T}} u_2 \rrbracket \wedge \llbracket v_1 =_{\mathbf{T}} v_2 \rrbracket = q$ . Then

$$\llbracket u_1 \lceil q = u_2 \rceil q \rrbracket \wedge \llbracket v_1 \lceil q = v_2 \rceil q \rrbracket = 1.$$

By using Lemma 7.4.9,

$$\llbracket (u_1 + v_1) \lceil q = (u_2 + v_2) \rceil q \rrbracket = 1.$$

Hence,

$$q \leq \llbracket u_1 + v_1 = u_2 + v_2 \rrbracket.$$

Similarly,

$$q \leq \llbracket u_1 \cdot v_1 = u_2 \cdot v_2 \rrbracket.$$

□

**THEOREM 7.4.12.** If  $B$  is a complete Boolean sublattice of  $Q(\mathcal{H})$ ,  $u$  is quantum real in  $V^B$ , and  $p \in B$ , then

$$p \leq \llbracket u \cdot \widehat{p} =_{\mathbf{T}} u \rrbracket, \quad \text{and} \quad p^\perp \leq \llbracket u \cdot \widehat{p} =_{\mathbf{T}} \check{0} \rrbracket.$$

*Proof.*  $p \leq \llbracket u =_{\mathbf{T}} u \rrbracket$  and  $p \leq \llbracket \widehat{p} =_{\mathbf{T}} \check{1} \rrbracket$ . Hence, by Theorem 7.4.11,

$$p \leq \llbracket u \cdot \widehat{p} =_{\mathbf{T}} u \cdot \check{1} \rrbracket \quad \text{and} \quad \llbracket u \cdot \check{1} = u \rrbracket = 1.$$

Therefore,  $p \leq \llbracket u \cdot \widehat{p} =_{\mathbf{T}} u \rrbracket$ .

$p^\perp \leq \llbracket u =_{\mathbf{T}} u \rrbracket$  and  $p^\perp \leq \llbracket \widehat{p} =_{\mathbf{T}} \check{0} \rrbracket$ . Hence, by Theorem 7.4.11,

$$p \leq \llbracket u \cdot \widehat{p} =_{\mathbf{T}} u \cdot \check{0} \rrbracket \quad \text{and} \quad \llbracket u \cdot \check{0} = \check{0} \rrbracket = 1.$$

Therefore,  $p^\perp \leq \llbracket u \cdot \widehat{p} =_{\mathbf{T}} \check{0} \rrbracket$ .

□

**THEOREM 7.4.13** ([11]). *If  $u$  and  $v$  are mutually compatible quantum reals in  $V^{Q(\mathcal{H})}$ , corresponding to self-adjoint operators  $A_u, A_v$ , respectively,*

$$A_u = \int \lambda dE_u(\lambda) \quad A_v = \int \lambda dE_v(\lambda),$$

*then*

$$A_{u+v} = \int \lambda dE_{u+v}(\lambda) = \int \lambda dE_u(\lambda) + \int \lambda dE_v(\lambda) = A_u + A_v$$

$$A_{u \cdot v} = \int \lambda dE_{u \cdot v}(\lambda) = \int \lambda dE_u(\lambda) \cdot \int \lambda dE_v(\lambda) = A_u \cdot A_v$$

### 7.4.3 Quantum complexes in $V^{Q(\mathcal{H})}$

$\{\vec{e}_j\}_{j \in J}$  continues to be a fixed orthonormal basis of  $\mathcal{H}$ , and  $p_j$  be the projection on  $\vec{e}_j$  for each  $j \in J$ . Let  $B$  be the complete Boolean algebra generated by orthogonal projections  $\{p_j \mid j \in J\}$ .

Since  $e^{i(\pi/2)I}$  is a unitary operator such that  $e^{i(\pi/2)I}(\vec{x}) = i\vec{x}$ ,

$$\{p_j\}_{j \in J} \cup \{e^{i(\pi/2)I}p_j\}_{j \in J}$$

is a set of mutually compatible projections in  $B$ .

**DEFINITION 7.4.8.**  $e^{i(\pi/2)I}\mathbf{1}$  is denoted by  $i$ , and  $e^{i(\pi/2)I}u$  for  $u \in \mathfrak{R}^{Q(\mathcal{H})}$  is denoted by  $i u$ .

$$e^{i(\pi/2)I}u \stackrel{\text{def}}{=} i u.$$

**DEFINITION 7.4.9.** If  $u, v$  are compatible quantum reals, then  $u + i v$  representing the pair  $\langle u, v \rangle$  is called a **quantum complex**.

$$u + i v \text{ is a complex number } \stackrel{\text{def}}{\iff} u \in \mathfrak{R} \wedge v \in \mathfrak{R} \wedge u \circ v$$

The set of all quantum complexes in  $V^{Q(\mathcal{H})}$  is denoted by  $\mathfrak{C}^{Q(\mathcal{H})}$ .

$$\mathfrak{C}^{Q(\mathcal{H})} = \{u + i v \mid u \in \mathfrak{R}^{Q(\mathcal{H})} \wedge v \in \mathfrak{R}^{Q(\mathcal{H})} \wedge u \circ v\}.$$

If  $u_1, v_1, u_2, v_2$  are compatible quantum reals, then

$$(u_1 + i v_1) + (u_2 + i v_2) \stackrel{\text{def}}{=} (u_1 + u_2) + i (v_1 + v_2)$$

$$(u_1 + i v_1) \cdot (u_2 + i v_2) \stackrel{\text{def}}{=} (u_1 \cdot v_1 - u_2 \cdot v_2) + i (u_1 \cdot v_2 + u_2 \cdot v_1)$$

$$(u_1 + i v_1) \circ (u_2 + i v_2) \stackrel{\text{def}}{\iff} \circ \{u_1, v_1, u_2, v_2\}, \quad i.e.$$

$\{u_1, v_1, u_2, v_2\}$  is a mutually compatible set.

If  $B$  is a maximal compatible subset of  $Q(\mathcal{H})$  and  $u, v$  are real numbers in  $V^B$ , then  $u + iv \in V^B$ .

#### 7.4.4 Compact complex numbers in $V^{Q(\mathcal{H})}$

A quantum complex in  $V^{Q(\mathcal{H})}$  represents a normal operator  $u$  on  $\mathcal{H}$ .

If  $\{q_j\}_{j \in J}$  is a complete orthogonal system in  $Q(\mathcal{H})$ , then there exists  $\sigma \in \mathcal{U}$  such that each  $q_j$  is the projection on  $\sigma(\vec{e}_j)$ .

$$q_j(\sigma(\vec{e}_k)) = \delta_{j,k} \sigma(\vec{e}_k) \quad \text{for } j, k \in J.$$

Each  $q_j$  will be represented in  $V^{Q(\mathcal{H})}$  by a real number  $\hat{q}_j$ .

$$q_j \leq \llbracket \hat{q}_j =_{\mathbf{T}} \check{1} \rrbracket = \llbracket \hat{q}_j =_{\mathbf{T}} \check{0} \rrbracket^{\perp}.$$

**LEMMA 7.4.14.** *If  $\{a_j\}_{j \in J} \subset \mathbb{R}$  and a complete orthogonal system  $\{q_j\}_{j \in J}$  in  $Q(\mathcal{H})$ , then there exists  $u$  in  $V^{Q(\mathcal{H})}$  such that*

$$q_j \leq \llbracket u =_{\mathbf{T}} \check{a}_j \rrbracket \quad \text{for each } j \in J.$$

*It follows that  $\llbracket u \in_{\mathbf{T}} \check{\mathbb{R}} \rrbracket = 1$ .*

*Conversely, if  $u \in V^{Q(\mathcal{H})}$  satisfies  $\llbracket u \in_{\mathbf{T}} \check{\mathbb{R}} \rrbracket = 1$ , then there exists  $\{a_j\}_{j \in J} \subset \mathbb{R}$  and a complete orthogonal system  $\{q_j\}_{j \in J}$  such that*

$$q_j \leq \llbracket u =_{\mathbf{T}} \check{a}_j \rrbracket \quad \text{for each } j \in J.$$

*Proof.* If  $\{a_i\} \subset \mathbb{R}$  and  $\{q_i\}_{i \in I}$  is a complete orthogonal system, then quantum real  $u$  in  $V^{Q(\mathcal{H})}$  defined by

$$\mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\}, \quad u(\check{r}) = \bigvee \{q_j \mid a_j \leq r\}$$

satisfies  $q_j \leq \llbracket u =_{\mathbf{T}} \check{a}_j \rrbracket$ . Therefore, there exists  $u$  such that  $q_j \leq \llbracket u =_{\mathbf{T}} \check{a}_j \rrbracket$ .

Conversely, if a quantum real  $u$  in  $V^{Q(\mathcal{H})}$  satisfies  $\llbracket u \in_{\mathbf{T}} \check{\mathbb{R}} \rrbracket = 1$ , then

$$\llbracket u \in_{\mathbf{T}} \check{\mathbb{R}} \rrbracket = \bigvee_{a \in \mathbb{R}} \llbracket u =_{\mathbf{T}} \check{a} \rrbracket = 1.$$

$\{\llbracket u =_{\mathbf{T}} \check{a} \rrbracket \mid a \in \mathbb{R}\}$  is compatible and for  $a \neq b$ ,

$$\llbracket u =_{\mathbf{T}} \check{a} \rrbracket \wedge \llbracket u =_{\mathbf{T}} \check{b} \rrbracket \leq \llbracket \check{a} =_{\mathbf{T}} \check{b} \rrbracket = 0.$$

Hence,  $\{\llbracket u =_{\mathbf{T}} \check{a} \rrbracket \mid a \in \mathbb{R}\}$  is a mutually orthogonal set. Therefore, there exists  $\{a_j\}_{j \in J} \subset \mathbb{R}$  and mutually orthogonal system  $\{q_j\}_{j \in J}$  such that

$$q_j \leq \llbracket u =_{\mathbf{T}} \check{a}_j \rrbracket.$$

□

Immediately from the preceeding lemma we have the following theorem.

**THEOREM 7.4.15.** *If  $\alpha_j = \{a_j + i b_j\} \subset \mathbb{C}$  and  $\{q_j\}_{i \in I}$  is a complete orthogonal system, then there exists  $u$  in  $V^{Q(\mathcal{H})}$  such that*

$$q_j \leq \llbracket u =_{\mathbf{T}} \check{a}_j + i \check{b}_j = \check{\alpha}_j \rrbracket.$$

Hence,  $\llbracket u \in_{\mathbf{T}} \check{\mathbb{C}} \rrbracket = 1$ .

Conversely, if  $u$  is a quantum complex in  $V^{Q(\mathcal{H})}$  such that  $\llbracket u \in_{\mathbf{T}} \check{\mathbb{C}} \rrbracket = 1$ , then there exists  $\{\alpha_i\}_{i \in I} \subset \mathbb{C}$  and a complete orthogonal system  $\{q_j\}_{j \in J}$  such that

$$q_j \leq \llbracket u =_{\mathbf{T}} \check{\alpha}_j \rrbracket.$$

**DEFINITION 7.4.10.** *For  $\{a_j\} \subset \mathbb{R}$  and a complete orthogonal system  $\{q_j\}_{j \in J}$ , the quantum real  $u$  in  $V^{Q(\mathcal{H})}$  such that  $q_j \leq \llbracket u =_{\mathbf{T}} \check{a}_j \rrbracket$  for each  $j \in J$  is denoted by  $\sum_{j \in J} \check{a}_j \hat{q}_j$ .*

$$u = \sum_{j \in J} \check{a}_j \hat{q}_j \stackrel{\text{def}}{=} \forall j \in J (q_j \leq \llbracket u =_{\mathbf{T}} \check{a}_j \rrbracket).$$

**DEFINITION 7.4.11.** *A quantum complex  $u$  is said to be **compact** if  $\llbracket u \in_{\mathbf{T}} \check{\mathbb{C}} \rrbracket = 1$ .*

$$u \text{ is a compact complex number} \iff \llbracket u \in_{\mathbf{T}} \check{\mathbb{C}} \rrbracket = 1$$

Remark. If  $\{\hat{q}_j\}_{i \in I}$  is a complete orthogonal system, then  $\{\check{a} \mid a \in \mathbb{C}\} \cup \{\hat{q}_j\}_{j \in J}$  is a compatible set of quantum complexes.

**COROLLARY 7.4.16.** *For  $\{a_j\}_{j \in J}, \{b_j\}_{j \in J} \subset \mathbb{C}$  and a complete orthogonal system  $\{q_j\}_{j \in J}$ ,*

$$\sum_{j \in J} a_j e_j = \sum_{j \in J} b_j e_j \iff \llbracket \sum_{j \in J} \check{a}_j \hat{q}_j = \sum_{j \in J} \check{b}_j \hat{q}_j \rrbracket = 1.$$

### 7.4.5 Quantum complexes in the sheaf representation

Quantum complexes represent normal operator valued functions on  $\mathcal{U}$ .

Let  $u + iv$  be a quantum complex.  $\{\llbracket \tilde{r} \in u \rrbracket \mid r \in \mathbb{Q}\} \cup \{\llbracket \tilde{r} \in v \rrbracket \mid r \in \mathbb{Q}\}$  is a compatible subset of  $Q(\mathcal{H})$ . Hence, there exists a unitary operator  $\sigma_0 \in \mathcal{U}$  such that

$$(\{\llbracket \tilde{r} \in u \rrbracket \mid r \in \mathbb{Q}\} \cup \{\llbracket \tilde{r} \in v \rrbracket \mid r \in \mathbb{Q}\}) \circ \{\sigma_0(p_j) \mid j \in J\}.$$

Let

$$E_u : \mathbb{R} \rightarrow Q \quad \text{by} \quad \bigwedge_{\lambda \leq r} \llbracket \tilde{r} \in u \rrbracket$$

$$E_v : \mathbb{R} \rightarrow Q \quad \text{by} \quad \bigwedge_{\mu \leq r} \llbracket \tilde{r} \in v \rrbracket$$

Then  $E_u$  and  $E_v$  are resolutions of identity and

$$\hat{u} = \int \lambda E_u(\lambda) \quad \hat{v} = \int \lambda E_v(\mu)$$

$\hat{u} + i\hat{v}$  is a normal operator on  $\mathcal{H}$ , where

$$\lambda_j = \inf\{r \in \mathbb{Q} \mid \sigma_0(p_j) \leq \llbracket \tilde{r} \in u \rrbracket\} \in \mathbb{R},$$

$$\mu_j = \inf\{r \in \mathbb{Q} \mid \sigma_0(p_j) \leq \llbracket \tilde{r} \in v \rrbracket\} \in \mathbb{R}.$$

$$f(\sigma_0) = \{\lambda_j \sigma_0(p_j) + i \mu_j \sigma_0(p_j)\}_{j \in J}$$

For each  $\sigma \in \mathcal{U}$ ,  $f(\sigma) = (\sigma \sigma_0^{-1}) f(\sigma_0) (\sigma \sigma_0^{-1})^{-1}$  is a normal operator on  $\mathcal{H}$ . Then  $f$  is a normal operator valued continuous function on  $\mathcal{U}$ , which represents the quantum complex  $u + iv$ , and forms a sheaf of ring of normal operator valued functions over  $\mathcal{U}$ , i.e.

$$F(U) = \{f(\sigma) \mid \sigma \in U\} \text{ for } U \in \mathcal{O}(\mathcal{U})$$

form a sheaf  $\text{Sh}_{\mathcal{U}} \mathfrak{C}$ , where  $\mathfrak{C}$  is the set of complex numbers in a Boolean valued universe.

## 7.5 Propositional system

**DEFINITION 7.5.1.** For elements  $b, c$  of a complete lattice  $\mathcal{L}$ , if  $b \leq c$ ,

$$b \text{ covers } c \stackrel{\text{def}}{\iff} \forall x[(b \leq x \leq c) \rightarrow (x = b \text{ or } x = c)]$$

$$a \text{ is an atom} \stackrel{\text{def}}{\iff} a \text{ covers } 0$$

A lattice  $\mathcal{L}$  is said to be **atomic** if for every element  $b$  such that  $b \neq 0$  there exists at least one atom  $p$  such that  $p \leq b$ .

**Propositional system** is a complete orthomodular lattice (p.39)  $\mathcal{Q}$  satisfying axioms A1 and A2, i.e.

**C1**  $c^{\perp\perp} = c$ ,

**C2**  $c \vee c^{\perp} = 1, \quad c \wedge c^{\perp} = 0$ ,

**C3**  $b \leq c \implies c^{\perp} \leq b^{\perp}$ .

**P**  $(\psi \rightarrow \varphi) \rightarrow (\psi \circ \varphi), \quad \text{where } \psi \circ \varphi \stackrel{\text{def}}{\iff} (\psi \rightarrow (\psi \wedge \varphi) \vee (\psi \wedge \varphi^{\perp}))$

**A1** :  $\mathcal{Q}$  is atomic, i.e. for every element  $b \in \mathcal{L}$  such that  $b \neq 0$  there exists at least one atom  $p$  such that  $p \leq b$ :

**A2** : Let  $p \in \mathcal{Q}$  is an atom and  $b \in \mathcal{Q}$ .

If  $p \wedge b = 0$ , then  $p \vee b$  covers  $b$ .

**LEMMA 7.5.1.** For  $a \in \mathcal{Q}$ , let  $\bar{a}$  be the set of atoms  $\leq a$  :

$$\bar{a} \stackrel{\text{def}}{=} \{p : \text{atom} \mid p \leq a\}$$

Then

$$\bigcup_{i \in I} \bar{a}_i = \overline{\bigvee_{i \in I} a_i}, \quad (\bar{a})^c = \neg \bar{a}$$

**DEFINITION 7.5.2.** A **symmetry transformation** is a bijective mapping of the propositional system onto itself which preserves the least upper bound and the orthocomplementation, i.e. automorphism:

**Aut1**  $\sigma(\bigvee_i a_i) = \bigvee_i \sigma(a_i)$

**Aut2**  $\sigma(a^\perp) = (\sigma(a))^\perp$

If  $\mathcal{Q}$  is a propositional system with symmetry transformations  $\mathcal{U}$ , and  $B$  is a maximal compatible subspace of  $\mathcal{Q}$ , then  $\mathcal{Q}$  has a sheaf structure of Boolean lattice over  $\mathcal{U}$ :  $\text{Sh}_{\mathcal{U}}B$

Hilbert quantum set theory is developed on a universe whose truth value set is a propositional system, with symmetry transformations.

# Bibliography

- [1] Birkhoff, G., ‘Lattice Theory’, 3rd ed., *AMS* 1967.
- [2] Birkhoff, G. and J.von Neumann, ‘The logic of Quantum Mechanics’, *Ann. Math.* 37:823-843, 1936.
- [3] Gentzen, G., ‘Untersuchungen über das logische Schliessen’, *Mathematische Zeitschrift* 39:176-210, 405-431, 1934-5.
- [4] Maehara, S., ‘Eine Darstellung der intuitionistischen Logik in der klassischen’, *Nagoya Mathematical Journal*, vol.7(1954), pp.45-64.
- [5] McNeille, H., ‘Partially Ordered Sets’, *Trans. AMS* 42:416-60, 1937.
- [6] Piron, C., *Foundations of Quantum Physics*, W.A. Benjamin, Inc., Massachusetts, 1976.
- [7] Rasiowa, H. and Sikorski, R., *The Mathematics of Metamathematics*, Warszawa 1963
- [8] Sikorski, R., *Cartesian product of Boolean algebras*, *Fundamenta Mathematicae* 37: 25-54, 1950.
- [9] Funayama, N., *On imbedding infinitely distributive lattice completely isomorphically into Boolean algebras*, *Nagoya Math. J.* 15 pp. 71-81(1959),
- [10] Takano, M., ‘Strong Completeness of Lattice Valued Logic’, *Archive for Mathematical Logic* 41:497-505, 2002.
- [11] Takeuti, G., *Two Applications of Logic to Mathematics*, Iwanami and Princeton University Press, Tokyo and Princeton, 1978.

- [12] Takeuti, G., ‘Quantum Set Theory’, *Current Issues in Quantum Logic*, E. Beltrametti and B. C. van Frassen (eds), Plenum, New York, 303-322, 1981.
- [13] Titani, S., ‘Lattice Valued Set Theory’, *Arch. Math. Logic* 38:395-421, 1999.
- [14] Titani, S. ‘Completeness of global intuitionistic set theory’, *J. Symbolic Logic*, vol.62-2(1997)506-528
- [15] Titani, S., ‘A Completeness Theorem of Quantum Set Theory’, *Handbook of Quantum Logic and Quantum Structures: Quantum Logic*, K. Engesser, D. M. Gabbay and D. Lehmann (eds), Elsevier Science Ltd., 661-702, 2009.
- [16] Titani, S., ‘Sheaf completeness of quantum logic’, *Computer Communication and Collaboration* (Vol. 3, Issue 2, 2015) ISSN 2292-1028(Print) 2292-1036(Online) , 2015
- [17] 森三樹三郎、‘ 莊子 ’ 中央公論新社
- [18] Zhuangzi, ‘The Complete Works of Zhuangzi’ , Translated by Burton Watson. Columbia University Press NewYork 2013

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